A Stochastic and Pretopological Modeling Aerial Pollution of an Urban Area

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RESUME Dans cet article, nous proposons une modélisation de la pollution aérienne dans une zone urbaine. On a développé un modèle prétopologique stochastique spatial pour appréhender ce problème de pollution aérienne. Nous présentons les propriétés principales de ce modèle et leurs interprétations en terme de pollution. Nous terminons par une discussion sur les perspectives que peut apporter un tel modèle dans le cadre d’une simulation de la dynamique de ces phénomènes de pollution dans la ville de Ouagadougou.

ABSTRACT: In this paper, we propose a model of aerial pollution of an urban area, the city of Ouagadougou. We develop a mathematical model of the spatial area from the pollution point of view. This model is based both on the concept of pretopological structure and the concept of random sets. The reader who is not familiar with pretopological concepts can find out all he needs on the

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Website http://www.pretopology.net/. We review positive points of our modeling and problems remaining to be solved in view to implement efficient policies for fighting pollution. In perspective, we discuss about some perspectives of simulation with the help of Multi-Agents Systems.

**MOTS-CLES:** prétopologie, modèle aléatoire, modélisation, diffusion, pollution aérienne.

**KEYWORDS:** pretopology, random sets, diffusion, pollution.

1. Introduction

Due to an important human activity, the level of aerial pollution (SO2, CO, CO2... rejections) may reach values with strong consequences on the health of populations living in the area. These aerial pollution phenomena are quite complex, influenced by a lot of factors. So designing efficient policies against pollution requires two main knowledge:

- Knowledge on mechanisms of dissemination of pollutants in atmosphere,
- Knowledge on socio-economic phenomena which play a role in pollution.

Nowadays, dissemination models allow knowing pollutants concentration in any point of an area from ground measures. By comparing to given thresholds, it is possible to determine if the air quality is good enough. In case when maximal authorized concentrations are overpassed, an action must be performed for reducing pollutants dissemination. To be efficient, this action must be based on a clear and objective determination of different responsibilities. So, we propose a mathematical model enabling to structure the urban area from the pollution point of view. Pretopology is the mathematical theory which provides this model. Furthermore, to take into account the complexity and the variety of factors more or less under control, we introduce a stochastic modeling, leading to a stochastic pretopological structuring of the area.

2. Mathematical Model
A. Notations and definitions

Let us consider a geographic area $G$ in which we suppose there exist $n$ polluters emitters, let be $E$ the set of these emitters, $E = \{x_1, x_2, ..., x_n\}$. For any $i, i = 1, ..., n$, we define a correspondence $G_i$ which denotes the area which is polluted by emitter $x_i$. Such $G_i(w)$ are defined as follows:

$$\forall i, i = 1, ..., n, \Gamma_i = \{x, x \in G / f_i(x, \omega) \geq \alpha + \eta\}$$

where
- $f_i$ is a function from $G \times (W, A, p)$ into $\mathbb{IR}$ and $f_i(x, w)$ denotes the concentration of pollutant on point $x$, under the scenario $w$, emitted by $x_i$. $(W, A, p)$ is a probability space.
- $a$ is the maximal value under which we can consider that the area is not polluted.
- $h$ is a scalar which gives how precise is the measuring of pollutants concentrations.

In the following, we suppose that, for any $i$, $G_i(w)$ is a measurable correspondence from $(W, A, p)$ into $(G, B_G)$ where $B_G$ is the Borelian family of $G$, so $G_i(w)$ is considered as a random correspondence.

B. Pretopological Structure

For any subset $A$ of $G$, we put:

$$a(w, A) = A \cup \bigcup_{i, i \in A} \Gamma_i(w)$$

So, it is easily to prove that:

Proposition 1

The function $a(w, \cdot)$ is a pseudoclosure function which defines a $V_D$ pretopological structure on $G$.

Proof

Obvious result by definition, it’s sufficient to verify the definition of a $V_D$ pretopological structure on $G$.

Given a pseudoclosure function, $G$ is endowed with a pretopological structure as said before. It becomes interesting to search for interpreting what it means for a subset $A$ to be a closed subset or an open subset. The two following propositions give an answer to this question.

Proposition 2

A subset $A$ of $G$ is a closed subset if and only if
\[ A \supseteq \bigcup_{i/x_i \in A} \Gamma_i(\omega) \]

**Proof**
It is sufficient to note that A closed subset of G means \( a(w, A) = A \) to get the result.

So, to say that A is a closed subset of G means that pollutants emitters in A do not export their pollutants outside of A.

**Proposition 3**
A subset A of G is an open subset if and only if
\[
\forall i=1,2,..,n, \Gamma_i(\omega) \cap A \neq \emptyset \implies x_i \in A
\]

**Proof**
A open subset means that \( A^c \) (complementary of A in G) is a closed subset. So, \( A^c \supseteq \bigcup_{i/x_i \in A^c} \Gamma_i(\omega) \) which implies:
\[
\forall i=1,2,..,n, \Gamma_i(\omega) \cap A \neq \emptyset \implies x_i \in A
\]
Conversely, \( \forall i=1,2,..,n, x_i \in A^c \implies \Gamma_i(\omega) \cap A = \emptyset \)
\[
\iff \forall i=1,2,..,n, x_i \in A^c \implies \Gamma_i(\omega) \subseteq A^c
\]
\[
\implies A^c \supseteq \bigcup_{i/x_i \in A^c} \Gamma_i(\omega)
\]
This leads to A open subset of G.

Thus A is an open subset of G means that its elements cannot be submitted to an outside pollution.

Our model is aimed to determine responsibilities in pollution situations from ground measures. For that, it is sufficient to be able to retrieve, from these measures, an estimate of subsets \( G_i(w) \) of G, as illustrated on the figure1.
For any $x$ in $G$, let us consider the set $J(x) = \{i, \ i=1,2,...,n \ / \ x \in G_i(w)\}$, then $\{x_i / i \in J(x)\}$ is the set of “direct” polluters of $x$ by definition. The following result gives the link between this last set and the fact a subset $A$ is open.

**Proposition 4**
If $A$ is an open subset of $G$, then $\{x_i / i \in J(x)\}$ is included in $A$.

**Proof**
Let $x$ in $A$ and let us suppose that $A$ is an open subset of $G$ and that $\{x_i / i \in J(x)\}$ is not included in $A$. Then, we get:

$\Rightarrow \exists i_0, x_{i_0} \notin A$ and $x \in \Gamma_{i_0}(\omega)$
$\Rightarrow \exists i_0, x_{i_0} \notin A$ and $x \in \Gamma_{i_0}(\omega) \cap A$
$\Rightarrow \exists i_0, x_{i_0} \notin A$ and $\Gamma_{i_0}(\omega) \cap A \neq \emptyset$

$\Leftrightarrow A$ is not an open subset
This leads to a contradiction, so we get the result.\[\square\]

The set of polluters of a point $x$ in $G$ may have pretopological properties. The following proposition establishes one of these properties.

**Proposition 5**
To say that $\{x_i / i \in J(x)\}$ is an open subset of $G$ is equivalent to the following assertion:
\[ \forall j, j = 1, 2, \ldots, n, \exists i, i \in J(x), x_i \in \Gamma_j(\omega) \Rightarrow x \in \Gamma_j(\omega) \]

**Proof**

Let A = \{x_i / i \in J(x)\}, A open subset of G means that A^c is a closed subset of G which is equivalent to:

\[ \forall j, j = 1, \ldots, n, (x_i \notin A^c) \Rightarrow \Gamma_j(\omega) \subseteq A^c \]

\[ \Leftrightarrow \forall j, j = 1, \ldots, n, (x_i \notin \Gamma_j(\omega)) \Rightarrow \forall i, i \in J(x), x_i \notin \Gamma_j(\omega) \]

\[ \Leftrightarrow \forall j, j = 1, \ldots, n, (\exists i, i \in J(x), x_i \notin \Gamma_j(\omega)) \Rightarrow x \notin \Gamma_j(\omega) \]

This leads to the result.

Thus, if the set of direct polluters of x is an open subset of G, we can say that the pollution phenomenon is a transitive one.

**Proposition 6**

If A is an open subset of G and if x belongs to A, then the set \{x_k / \exists i, i \in J(x), x_i \in G_k(w)\} is included in A.

**Proof**

Let us suppose that A is an open subset of G and that the set \{x_k / \exists i, i \in J(x), x_i \in G_k(w)\} is not included in A. Then:

\[ \exists k_0, \exists i, i \in J(x), x_i \in \Gamma_{k_0}(\omega) \text{ and } x_k \notin A. \]

But A is an open subset of G, so \[ x_k \in A \Rightarrow \Gamma_k(\omega) \subseteq A^c, \]

then we get that \[ \exists k_0, \exists i, i \in J(x), x_i \notin \Gamma_k(\omega). \]

As \[ i \in J(x), \Gamma_k(\omega) \cap A = \emptyset, \]

and \[ x_i \in A \text{ because } A \text{ is an open subset. This leads to a contradiction, then to the result.} \]

Now, we build the finite following sequence:

\[ I_0(x) = J(x) \]

\[ \forall p, p \geq 1, I_p(x) = I_{p-1}(x) \cup \{k / \exists i, i \in J_{p-1}(x), x_i \in \Gamma_k(\omega)\} \]

As, we supposed there exists a finite number of polluters, we can say:

\[ \exists n_0, n_0 \in N, I_{n_0-1}(x) = I_{n_0}(x) \]

We denote S(x) this last set.

**Proposition 7**

The set \[ B(x) = \{x\} \cup \{x_i / i \in S(x)\} \] is the smallest open subset of G which contains x.
Proof
To prove that B(x) is an open subset of G, it is sufficient to prove that B'(x) is a closed one. Obvious. Let us consider A an open subset which contains x. Then for any x in J(x), x_i is an element of A and \{x_k / \exists i, i \in J(x), x_i \in G(x)\} is included in A. So, by definition of S(x), \forall i \in S(x), \Gamma_i(\omega) \cap A \neq \emptyset, then x_j belongs to A and B(x) is included in A.

These sets B(x) can be interpreted as a chain of pollution reaching x. Let's consider V(x) = \{x\} \cup \{x, i \in J(x)\}.
We get:

**Proposition 8**
\( V(x) \) is the smallest neighborhood of x.

**Proof**
\( V \) is a neighborhood of x if and only if \( x \in i(w, V) \) where \( i(w, .) \) is the pretopological interior associated with \( a(w, .) \) defined by \( i(w, A) = a(w, A^e) \). Moreover, we know that (see [1]) if \( V_x \) denotes the set of neighborhoods of x, \( a(w,A) \) can be expressed as:
\[
\begin{align*}
a(w, A) = & \{ x \in G / \forall V, V \in V_x, V \cap A \neq \emptyset \} \\
\end{align*}
\]
Two cases must be considered according to \( \{x, i \in J(x)\} \) is empty or not. If \( \{x, i \in J(x)\} \) is empty, \( V(x) = \{x\} \) and \( i(w, \{x\}) = \{x\} \). This leads to the result. Else, let \( x \in a(\omega, V^{c}(x)) \)
\[
\begin{align*}
\Leftrightarrow & \ x \in \{x, i \in J(x)\}^2 \cup \bigcup_{i \in J(x)} \Gamma_i(\omega) \\
\Leftrightarrow & \ x \in V(\omega) \text{ ou } \exists i = 1, 2, \ldots, n, x_i \notin V(x), x \in \Gamma_i(\omega) \\
\text{but} & \ x_i \notin V(x) \Rightarrow x \notin \Gamma_i(\omega). \\
\text{This leads to a contradiction and to the result.} \\
\end{align*}
\]

**Proposition 9**
\( G, a(w, .) \) is a separate pretopological space if and only if
\[
\forall x, x \in G, \forall y, y \in G, (x \neq y \Rightarrow V(x) \cap V(y) = \emptyset)
\]

**Proof**
It is sufficient to apply the definition of a pretopological separate space. 

This last proposition shows that the pretopological space \( G, a(w, .) \) is a separate one if and only if two any of its points never have a common direct polluter and they are not direct polluter of each other.
Given any pretopological space of V type (then of V_D type), we can define a preconvergent space by defining for any x the family Q(x) of prefilters which are strictly finer than the prefilter V_x of its neighborhoods. Then, concepts of points adherent to a prefilter and of points limits of a prefilter. This enables us to get the two following results which are direct consequences of the definitions.

**Proposition 10**
A point y of G is a point adherent to the prefilter V_x if and only if

\[ y \in \bigcup_{i \in I(x)} \Gamma_i(\omega) \]

**Proposition 11**
If y is a direct polluter of x, then y is a point limit of V_x.

### 3. Mesurability and integrability problems

The previous section was aimed to present basic pretopological properties of space G. For that, we worked with a fixed w in (W, A, p). In fact, we have to define and study pretopological spaces for any possible w. In the following, we suppose that (W, A, p) is a complete probability space, W being a locally compact set and A a s-algebra on W. When talk about measurability of correspondences. It must be noted that, concerning measurability, G is considered as a topological space due to the fact concepts of measurability are not yet generalized to pretopological spaces. Then, we can say:

**Proposition 12**
If for any I, i=1,2,..,n, correspondences G_i(w) are measurable from (W,A,p) into (G,B_G), then the correspondence a(_,A) from (W,A,p) into (G,B_G) is also measurable for any A subset of G.

**Proof**
Proof straightforward by noting the union of measurable correspondences is measurable.

A measurable correspondence is called a random correspondence. Aumann proposed an integrability concept for random correspondences we use in this paper. This leads to the following result:
Proposition 13
If for any $i=1,2,\ldots,n$, correspondences $G_i(w)$ are integrable according to the Aumann’s definition, then the random correspondence $a(\cdot,A)$ (for any $A$ subset of $G$) is also Aumann integrable.

Proof
Immediate.

We put $\forall A, A \subset G, \bar{a}(A) = \int a(\omega,A)P(d\omega) = E(a(\omega,A))$

Then:

Proposition 14
The pretopology defined by $\bar{a}(\cdot)$ on $G$ is of $V$ type.

Proof
$\bar{a}(\varnothing) = E(a(\omega,\varnothing)) = E(\varnothing) = \varnothing$
$\forall \omega, \omega' \in \Omega, A \subset a(\omega,A)$
$\Rightarrow A \subset E(a(\omega,A))$
$\Leftrightarrow A \subset \bar{a}(A)$
$\forall \omega, \omega' \in \Omega, A \subset B \Rightarrow a(\omega,A) \subset a(\omega,B)$
$\Rightarrow A \subset B \Rightarrow E(a(\omega,A)) \subset E(a(\omega,B))$
$\Leftrightarrow A \subset B \Rightarrow \bar{a}(A) \subset \bar{a}(B)$

Although for any $w$ in $W$, $a(w,\cdot)$ defines a $V_D$ pretopological structure, only defines a $V$ pretopological structure. This is due to the fact that the integral of the union of two random correspondences is not the union of their integrals in general.

Proposition 15
The correspondence $T_x$ from $(W,A,p)$ into $G$ defined by:
$\forall x \in G, \omega \mapsto T_x(\omega) = V(x)$, where $V(x)$ is the smallest neighborhood of $x$, is measurable and Aumann integrable.
Proof
Let \( x \) in \( G \) and \( i \) in \( \{1,2,\ldots,n\} \). Let put \( F_i \), subset of \( W \) defined by
\[
F_i = \{ \omega / \Omega_i(\omega) \cap \{x\} \neq \emptyset \}. \]
\( F_i \) is an element of \( A \). Let \( A^+_i = \{ \{x\}, \text{if } \omega \in F_i \}
\[
\emptyset \text{, else}
\]
Let \( F \) a closed subset of \( G \), we put \( A = \{ \omega \in \Omega / A^+_i \cap F \neq \emptyset \} \). Two cases can occur:

\( x_i \in F \), then \( A = F_i \), then \( A \in A \),
\( x_i \notin F \), then \( A = \emptyset \), then \( A \in A \).

Thus \( A^+_i \) is measurable. But \( V(x) = \{ \{x\} \cup \bigcup_{i=1}^{n} A^+_i(\omega) \in T_x \} \), which ensures that \( T_x \) is measurable.

Let us consider the function \( g(.) \) from \( (W,A,p) \) into \( G \) defined by \( g(w) = x \), we can see that obviously it is an integrable selection of \( T_x \). So \( T_x \) is Aumann integrable.

From this last result, it is possible to define the “average” of the smallest neighborhoods of \( x \). Moreover, we say that a function \( f_i \) is locally constant in \( w \) if and only if there exists a neighborhood of \( w \) in which \( f_i \) takes a same value. Then, it is possible to prove the two following results. The first one gives a condition for \( G_i \) is a half lower continue random correspondence from \( W \) into \( (G,\bar{a}) \).

**Proposition 16**
If the function \( f_i \) is locally constant in any element of \( W \), then the random correspondence \( G_i \) is half lower continue from \( W \) into \( (G,\bar{a}) \).

The second one gives a condition for \( G_i \) is a half lower continue random correspondence from \( W \) into \( (G,\bar{a}) \).

**Proposition 17**
If the function \( f_i \) is locally constant in any element of \( W \), then the random correspondence \( G_i \) is half upper continue from \( W \) into \( (G,\bar{a}) \).

4. Perspectives : simulations

According to this model, the next step following the modeling of this complex phenomena, it’s simulation. In a simulation, we will attempt to reconstruct the scenario of an urban air pollution
deduced from the data collection available from various databases (meteo, amma, health, ...) integrating the concepts presented in the previous section. We’ll propose an multi-agent based framework which appear to be the best method to play the scenario and to implement the model’s characteristics. This simulation work should indicate the chain of polluters and their responsibilities in air pollution under the various scenario as well as allowing the prediction of other risks that would occurred under some hypothesis.

This section will first emphasizes on the MAS simulation advantages and then its architecture will be sketched.

A. Generalities about complex systems simulation

In various natural and artificial contexts, we observe phenomena of great complexity. However, research in physics, biology and in other scientific fields showed that the elementary components of complex systems are quite simple. It became crucial for scientific research dealing with complex systems to determine the mathematical mechanisms allowing including and understanding how a certain number of such elementary components, acting together, can produce the complex behaviors observed in these systems.

The second law of thermodynamics is an example of a general principle describing the global behavior of different kind of systems. It implies that the initial order is gradually degraded while a system evolves over time, so that at the end a state of maximum disorder (i.e. entropy) is reached. Many natural systems exhibit such a behavior. But there are also various systems having an opposite behavior; initial simplicity or disorder is transformed into great organized complexity.

B. The multi-agent modeling and simulation paradigm

An agent is an entity that acts in its environment and executes autonomous actions to reach certain goals. The environment where it evolves is not necessarily deterministic. Agent-based systems are of increasing importance. They are regarded as a new paradigm enabling an important step forward in empirical sciences, technology and theory [12].

Every agent is characterized by autonomous behavior. This consists in the first instance of the so-called inherent dynamism that an intelligent agent displays autonomously without input from the environment. In
addition there is the induced dynamism, which describes how the intelligent agent reacts in response to inputs from the environment. Some specific "intelligent" agents can exchange information with their environment and with other intelligent agents. By means of the possibility of communication an intelligent agent must obtain information about its environment, which enables it to build up its own world model. The Modeling and Simulation of real systems consisting of intelligent agents that cooperate with each other has recently emerged as an important field of research. Systems of this kind can be found in the empirical sciences, in technology and in mathematical theories. Cooperation between intelligent agents can produce a stable system that displays new global behavior on the next higher level of abstraction. The task is to explain this global behavior on the basis of the individual attributes of the intelligent agents and of their interactions.

C. A multi-agent approach: motivations

Multi-agent systems (MAS) represent an attempt to design the simplest model able to generate a great complexity. MAS are microscopic models for complex natural systems containing large numbers of different components with local interactions. Even if the construction of a MAS based model is very simple, their behavior can be very complex. There are fundamental reasons showing that there is no general method, which can universally be applied to predict the behavior of these systems. Compared with reality the MAS appear simplistic. However, they are currently considered as a fundamental tool in simulating complex phenomena, in particular concerning the auto-organized systems. The use of agents makes it possible to reduce the complexity of modeling to what is necessary to generate the phenomenon. It is a paradox of complex systems: the behavior of the system is unpredictable and complex (at a long term level) whereas the laws (or rules) controlling it are simple and deterministic. Moreover a MAS represent a powerful simulation tool. In fact a good simulation supposes computing power which can be obtained through parallel computing. The non-local nature of interactions between agents makes the programming of cellular automata easy to be parallelized.

MAS can be regarded as a discrete representation of the partial derivative equations modeling the studied complex phenomenon. They are also considered as a computational frame work to implement the mathematical model of the physical system.
The main advantages of the MAS based simulations:

- it allows a realistic simulation, using parallel and distributed simulation,
- it allows the integration of local data and geometrical information about the environment and the active agents.
- It allows relaxing constraints and adapting and individualize the behavior of different agents.
- Allows the heterogeneity in the system (behaviors, entities…).

**D. A multi-agent simulation for air pollution**

Our objective will be to develop a multi-agent based simulation based on the following aspects (representing the different agents of the system):

- Representation of the space using a square lattice.
- Representation of active entities (emitters of pollution).
- Representation of the environmental and climatic conditions.

The simulation will be first designed to test our modeling, we will indeed consider GIS and agent-based simulation when data will be available. For the sake of illustration, one can easily map the problem onto a square lattice.

Agents will be of different types:

- fixed emitters of pollutant emitters (factories, houses)
- mobile pollutants emitters (cars, motorbike,…) 

These pollutants emitters will be agent placed on some location of the square lattice. They will emit pollution with a certain strength, but the pollution will decrease with distance.

Agents bringing information to the simulation will be the cells of the square lattice, submitted to pollution, their main task will be to compute their level of pollution. Open and closed subsets of the modeling will be outputted through the pseudo-closure computation, thus defining region $A_k$.

Weather conditions will be agents defining the spreading of pollutants

A regulation agent could embody the public policy for regulating such or such possible action of the state.
E. Implementation tool


While the classical modules implemented are based on the dialog between cells, this model we need also methods based on the pretopological concepts (i.e. pseudo-closure, closure).

5. Conclusion

As a conclusion, we can say that the stochastic/pretopological approach seems adequate for modeling aerial pollution diffusion. It enables to make the link between ground measures and the mathematical model via two main ideas.

The first one consists in considering particular random correspondences, say “A simple correspondences”. It corresponds to the idea that the measuring system permits to detect areas in which the pollutant level is constant, higher than the admissible threshold, and to integrate the fact that the pollution can decrease according to a given direction until to be considered vanishing. Thus, the random correspondence can be written

$$
\Gamma_i(\omega) = \bigcup_{j=1}^{p} [K_j \cap I_{A_k}]$$

where $K_j$ is a subset of $G$ on which the level of pollutant is constant, equal to $l_j$, and $I_{A_k}$ denotes the correspondence defined by $I_{A_k} = G$ if $\omega \in A_k$ and $I_{A_k} = \emptyset$ if $\omega \notin A_k$.

The second one consists in exploiting ground measures by mean of estimation theory for random correspondences (see [6]).

For the future work, we aim to enrich the model with other local behaviors and plan to interpret it with the real data. This link between observations and model becomes possible to feed a simulation model based on multi agents systems. The implemented model will allow the exhibition of the phase transition phenomenon observed around a critical value of the initial density of pollution emitters.

This kind of tool is very helpful along the decision process.
Bibliographie


