Variable Neighborhood Search for Extremal Graphs. 24. Results about the clique number

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RÉSUMÉ. Un ensemble de sommets $S$ d’un graphe $G$ est une clique si chaque paire de ses sommets est reliée par une arête. La cardinalité maximum d’une clique, notée $\omega$, est le nombre de sommets d’une plus grande clique dans $G$. Une série de bornes inférieures et supérieures sur la différence, la somme, le quotient ou le produit de $\omega$ et quelques invariants graphiques usuels, est obtenue par le système AGX 2, dont la plupart sont prouvées soit automatiquement soit à la main. Dans cet article, on présente de telles bornes inférieures et supérieures en prenant, comme second invariant, les degrés minimum, moyen et maximum, les connectivités algébrique aux sommets et aux arêtes, l’index, la distance moyenne, l’éloignement, le rayon et le diamètre.

ABSTRACT. A set of vertices $S$ in a graph $G$ is a clique if any two of its vertices are adjacent. The clique number $\omega$ is the maximum cardinality of a clique in $G$. A series of best possible lower and upper bounds on the difference, sum, ratio or product of $\omega$ and some other common invariants of $G$ were obtained by the system AGX 2, and most of them proved either automatically or by hand. In the present paper, we report on such lower and upper bounds considering, as second invariant, minimum, average and maximum degree, algebraic, node and edge connectivities, index, average distance, remoteness, radius and diameter.

MOTS-CLÉS : Cardinalité maximum d’une clique, Invariant, Graphe extrême, AGX
KEYWORDS: Clique number, Invariant, Extremal graph, AGX

Studia Informatica Universalis.
1. Introduction

In the thesis [1] (see also [3] for a summary) a systematic comparison was made on relations between pairs of graph invariants selected from a set of 20, which includes the clique number. The general form of these relations, called AGX Form 1, is

\[ b(n) \leq i_1 \oplus i_2 \leq \overline{b}(n) \]

where \( i_1 \) and \( i_2 \) are graph invariants, \( \oplus \) denotes one of the four operations \(-, +, /, \times\), \( b(n) \) and \( \overline{b}(n) \) lower and upper bounding functions for \( i_1 \oplus i_2 \) depending on the order \( n \) (or number of vertices) of the graphs under consideration. These bounding functions are requested to be best possible in the strong sense, i.e., for all \( n \) (except possibly for very small values due to border effects) there is a graph such that the lower (upper) bound is attained. In the present paper, we focus on some of the results of that study, which concern the clique number. More precisely we study relations of the form

\[ b(n) \leq \omega \oplus i \leq \overline{b}(n), \]  \hspace{1cm} (1)

where \( \omega \) is the clique number and \( i \) is minimum, average and maximum degree, algebraic, node and edge connectivity, index, average distance, remoteness, radius or diameter.

The relationships of the form (1) discussed and proved in this paper were obtained, as well as many others [1, 3, 6], with the system AutoGraphiX 2 (AGX 2, for short) [2]. This “discovery system” is described, together with its results, in a series of papers, to which the present one also belongs, under the common title “Variable Neighborhood Search for Extremal Graphs”. It is based on the following observation: a large variety of problems in extremal graph theory can be viewed as parametric combinatorial optimization ones defined on the family of all graphs (or some restriction thereof) and solved by a generic heuristic. The parameter is usually the order \( n \) of the graphs considered (sometimes the order \( n \) and the size \( m \)). The heuristic fits in the Variable Neighborhood Search metaheuristic framework [8, 14, 15]. Presumably extremal graphs are found by performing a series of local changes (removal, addition or rotation of an edge, etc ...) until a local optimum is reached, then applying increasingly large perturbations, followed by new descents; if
a graph better than the incumbent one is found, the search is recentered there. After the parametric family of extremal graphs has been found, relationships between graph invariants may be deduced from them using various data mining techniques [9]. These include (i) a numerical method based on Principal Component Analysis which yields a basis of affine relations between the graph invariants considered; (ii) a geometric method which uses a gift-wrapping algorithm to find the convex hull of extremal graphs viewed as points in the invariant space; facets of this convex hull give inequality relations; (iii) an algebraic method which recognizes families of graphs then exploits a database of formulae giving expressions of invariants as functions of \( n \) on these families; substitution in (1) then leads to linear or nonlinear conjectures. As mentioned above, a systematic comparison of 20 invariants has been made in [1]. Results are summarized in [3], available in detail on the website “http://www.gerad.ca/~agx”, and currently being proved in a series of papers (see [4] for references).

The conjectures discussed and proved in the present paper were obtained using the numerical and, mostly, the algebraic method. Some easy conjectures were proved by the system exploiting the fact that relevant families of extremal graphs for individual invariants may (and quite often do) have non-empty intersection.

Let \( G = (V, E) \) be an undirected connected graph of order \( n = |V| \) and size \( m = |E| \). A clique in \( G \) is a set of vertices \( S \subset V \) such that every two vertices in \( S \) are adjacent. The number of vertices of the largest clique in a graph is called its **clique number** and denoted by \( \omega \). The **degree** \( d(v) \) of a vertex \( v \) in \( G \) is the number of its neighbors. The **minimum**, **average** and **maximum degrees** of \( G \) are denoted by \( \delta, \bar{d} \) and \( \Delta \) respectively. The vertex (resp. edge) **connectivity** of a graph \( G \), denoted by \( \nu \) (resp. \( \kappa \)), is the smallest number of vertices (resp. edges) whose deletion disconnects \( G \) or reduces it to a single vertex. The **algebraic connectivity** of \( G \), denoted by \( \alpha \), is the second smallest eigenvalue of the Laplacian of \( G \). The Laplacian is the matrix defined by \( L = \text{Diag} - A \), where \( \text{Diag} \) is the diagonal matrix of vertex degrees, and \( A \) is the adjacency matrix of \( G \). The index or **spectral radius** \( \lambda_1 \) of \( G \) is the largest eigenvalue of its adjacency matrix. The **distance** \( d(u, v) \) between two vertices \( u \) and \( v \) of \( G \) is the length of a shortest path connecting \( u \) and \( v \). The **average distance** between distinct vertices of \( G \) is denoted by \( \bar{d} \). The
transmission of a vertex $v$ in $G$ is the sum of all distances between $v$ and the other vertices. It is said to be normalized if divided by $n-1$. The remoteness is the maximum normalized transmission and is denoted by $\rho$. The eccentricity of a vertex $v$ in $G$ is the maximum over the distances from $v$ to the other vertices. The radius, denoted by $r$, is the smallest eccentricity. The diameter, denoted by $D$ is the largest eccentricity. The chromatic number $\chi$ of $G$ is the minimum number of colors to assign to the vertices of $G$ such that any two adjacent vertices have different colors. An independent set in $G$ is a vertex subset in which no two vertices are adjacent. The number of vertices of the largest independent set in a graph is denoted by $\alpha$ and called the independence number. The complete graph on $n$ vertices is denoted by $K_n$. The cycle on $n$ vertices is denoted by $C_n$. The kite $KT_{n,\omega}$ on $n$ vertices with clique number $\omega$, $2 \leq \omega \leq n$, is a graph consisting of a clique on $\omega$ vertices with an appended path on $n-\omega+1$ vertices. If $\omega = n-1$, the kite is short; if $\omega = 3$, the kite is long. A graph in which the vertex set can be partitioned into two independent sets is bipartite; three sets tripartite; $k$ sets $k$-partite or multipartite with $k$ independent sets. A $k$-partite graph is said to be complete if any two vertices are adjacent if and only if they belong to different partition classes. A $k$-partite graph is said to be balanced, and denoted by $T_k(n)$, if for any two partition classes $V'$ and $V''$, $||V'|-|V''|| \leq 1$. It is also called Turán’s graph. First, let us recall Turán’s theorem.

**Théorème 1.1 (Turán [17])**: If $G$ is a $K_{q+1}$ free graph on $n$ vertices with $m$ edges, then

$$m \leq \left(1 - \frac{1}{q}\right) \cdot \frac{n^2}{2}.$$ 

Equality holds if and only if $G$ is isomorphic to a complete $q$-partite graph in which all classes are of equal cardinality.

The remaining part of this paper is organized as follows: relations involving degrees, i.e., $\delta$, $\overline{d}$ and $\Delta$, are considered in Section 2. In Section 3, bounds involving algebraic, node and edge connectivities, i.e. $a$, $\nu$ and $\kappa$, are considered. We report on the bounds related to the index $\lambda_1$ in Section 4. In Section 5, bounds involving metric invariants, i.e., average distance $l$, remoteness $\rho$, radius $r$ and diameter $D$, are discussed.
Brief conclusions are drawn in Section 6. A table of all results discussed in this paper, mentioning also how they were proved, is given in the Appendix.

2. Clique Number and Degrees

2.1. Clique number and minimum degree

Among the 8 bounds involving the clique number \( \omega \) and the minimum degree \( \delta \), 4 are easy and AGX 2 proved them automatically. These cases correspond to the upper and lower bounds on \( \omega + \delta \) and on \( \omega \cdot \delta \), and are given in the summary of results table in the Appendix. In 3 cases, which correspond to the upper bound on \( \omega - \delta \) and both lower and upper bounds on \( \omega / \delta \), AGX 2 generated automatically conjectures which were proved by hand. In the last case, the lower bound on \( \omega - \delta \) was first obtained as a structural conjecture, and the corresponding formula derived by hand using the algebraic properties of the extremal graphs. These 4 conjectures are proved in Theorems 2.1 and 2.2 below.

**Theorem 2.1**: Let \( G = (V, E) \) be a connected graph on \( n \geq 3 \) vertices with minimum degree \( \delta \) and clique number \( \omega \). Then

\[
\left\lfloor \sqrt{n} \right\rfloor - \left\lfloor \left( 1 - \frac{1}{\sqrt{n}} \right)n \right\rfloor \leq \omega - \delta \leq n - 2.
\]

The lower bound is best possible as shown by balanced complete multipartite graphs \( T_{\left\lfloor \sqrt{n} \right\rfloor}(n) \) or \( T_{\left\lceil \sqrt{n} \right\rceil}(n) \). The upper bound is attained if and only if \( G \) is the short kite \( KT_{n,n-1} \) or \( K_3 \).

**Proof**: Lower bound: It is easy to see that equality holds for \( T_{\left\lfloor \sqrt{n} \right\rfloor}(n) \) and \( T_{\left\lceil \sqrt{n} \right\rceil}(n) \).

Using Turán’s theorem, recalled above, and the fact that \( \delta \leq \overline{d} = 2n/n \), we have

\[
\delta \leq \left\lfloor \left( 1 - \frac{1}{\omega} \right)n \right\rfloor.
\]
Hence
\[ \omega - \delta \geq \omega - \left( 1 - \frac{1}{\omega} \right) n = \omega - n + \left\lceil \frac{n}{\omega} \right\rceil \geq \omega - n + \frac{n}{\omega}. \]

Consider now the function \( f(x) = x - n + n/x \), for a fixed \( n \) and \( 2 \leq x \leq n \). It is easy to see that if \( x \) is a real number \( f(x) \) is minimum if and only if \( x = \sqrt{n} \). So, if \( x \) is an integer \( f(x) \) is minimum if \( x = \lfloor \sqrt{n} \rfloor \) or \( x = \lceil \sqrt{n} \rceil \), and as \( \omega - \delta \) is an integer the bound must be \( \min \{ \lceil f(\lfloor \sqrt{n} \rfloor) \rceil, \lceil f(\lceil \sqrt{n} \rceil) \rceil \} \). A substitution shows that \( \lceil f(\lfloor \sqrt{n} \rfloor) \rceil = \lceil f(\lceil \sqrt{n} \rceil) \rceil \). Thus the bound follows.

**Upper bound:** To prove this bound we consider three cases, depending on the values of \( \omega \).
- If \( \omega = n \), \( G \) is the complete graph \( K_n \) and \( \omega - \delta = 1 \). In this case the bound is reached if and only if \( G \) is \( K_3 \).
- If \( \omega = n - 1 \), then \( \omega - \delta \leq n - 2 \) with equality if and only if \( \delta = 1 \), i.e., \( G \) is the short kite \( KT_{n,n-1} \).
- If \( \omega \leq n - 2 \), then \( \omega - \delta \leq n - 3 \).
Thus the result follows. \( \Box \)

To prove the next theorem, we need the following preliminary result.

**Lemma 2.1:** Let \( G = (V,E) \) be a connected graph with an odd number \( n \) of vertices, clique number \( \omega = 2 \) and minimum degree \( \delta = (n-1)/2 \). Then \( G \) is a complete balanced bipartite graph \( T_2(n) \) or the cycle \( C_5 \).

**Proof:**
The case \( n \leq 3 \) is trivial, so assume that \( n \geq 5 \). Let \( v \) be a vertex with \( d(v) = \delta = (n-1)/2 \), \( A = \Gamma(v) \) the set of neighbors of \( v \) and \( B = V \setminus (A \cup \{v\}) \). Note that \( |A| = |B| = (n-1)/2 \). Since \( \omega = 2 \), two vertices from \( A \) cannot be adjacent to each other. Then each vertex from \( A \) has at least \( (n-1)/2 - 1 \) neighbors in \( B \).
Now, if there is no edge between vertices from \( B \) then each vertex in \( B \) is adjacent to all vertices from \( A \), and thus \( G \) is \( T_2(n) \).
If there is an edge between two vertices in \( B \), say \( v_1 \) and \( v_2 \), let \( d_A(v_i) \)
denote the number of neighbors of \( v_i, i = 1, 2, \) in \( A \). Since \( \omega = 2 \), \( v_1 \) and \( v_2 \) have no common neighbor (in \( A \)), and then
\[
d_A(v_1) + d_A(v_2) \leq \frac{n - 1}{2}.
\]
Since \( d(v_1) \geq \delta = (n - 1)/2 \) and \( |B| = (n - 1)/2 \), \( v_1 \) has at least a neighbor, say \( u \) in \( A \). So \( u \) has exactly \((n - 1)/2 - 1\) neighbors in \( B \), all the vertices in \( B \) except \( v_2 \). Thus \( v_1 \) has no neighbor in \( B \) except \( v_2 \). Then \( d(v_1) = d_A(v_1) + 1 \geq (n - 1)/2 \). Similarly \( d(v_2) = d_A(v_2) + 1 \geq (n - 1)/2 \). So we get
\[
2 \frac{n - 1}{2} - 2 = n - 3 \leq d_A(v_1) + d_A(v_2) \leq \frac{n - 1}{2},
\]
which is true (since \( n \geq 5 \)) if and only if \( n = 5 \) and \( G \) is \( C_5 \).

**Théorème 2.2**: Let \( G = (V, E) \) be a connected graph, \( G \neq K_3 \), on \( n \geq 3 \) vertices with clique number \( \omega \) and minimum degree \( \delta \). Then
\[
\frac{2}{\left\lfloor \frac{n}{2} \right\rfloor} \leq \frac{\omega}{\delta} \leq n - 1.
\]
The lower bound is attained if and only if \( G \) is the balanced complete bipartite graph \( T_2(n) \), \( C_5 \) or \( T_3(9) \), and the upper bound if and only if \( G \) is the short kite \( KT_{n,n-1} \).

**Proof**:
Lower bound: From Theorem 1.1, it is easy to see that
\[
\delta \leq n - \left\lfloor \frac{n}{\omega} \right\rfloor
\]
and then
\[
\frac{\omega}{\delta} \geq \frac{\omega}{n - \left\lfloor \frac{n}{\omega} \right\rfloor}.
\]
So to prove the bound, it suffices to prove that
\[
\frac{\omega}{n - \left\lfloor \frac{n}{\omega} \right\rfloor} \geq 2 \left\lfloor \frac{\omega}{2} \right\rfloor.
\]
To be done, we consider two cases:
(i) Case $n = kw + l$, where $1 \leq l \leq \omega - 1$, or $n$ is even. Note that
\[
\left\lceil \frac{n}{\omega} \right\rceil \left( n - \left\lceil \frac{n}{\omega} \right\rceil \right) \leq \left\lceil \frac{n}{2} \right\rceil \left( n - \left\lceil \frac{n}{2} \right\rceil \right).
\] (2)

a) If $n$ is even, we have
\[
\omega \left\lceil \frac{n}{\omega} \right\rceil \geq 2 \left\lceil \frac{n}{2} \right\rceil
\] (3)

Using (2) and (3), we get
\[
\omega \left\lceil \frac{n}{\omega} \right\rceil \geq 2 \left\lceil \frac{n}{2} \right\rceil \geq 2 \left\lceil \frac{n}{2} \right\rceil = \frac{2}{n}.
\]

b) If $n$ is odd and $n = kw + l$, where $1 \leq l \leq \omega - 1$, we have
\[
\omega \left\lceil \frac{n}{\omega} \right\rceil = \omega (k + 1) = n + (\omega - l) \geq n + 1 = 2 \left\lceil \frac{n}{2} \right\rceil.
\] (4)

Thus, using (2) and (4), we get
\[
\frac{\omega}{n - \left\lceil \frac{n}{\omega} \right\rceil} \geq \frac{2}{n - \left\lceil \frac{n}{2} \right\rceil} = \frac{2}{n}.
\]

(ii) Case $n = \omega k$ and $n$ odd. Note that in this case $\omega$ is necessarily odd, so $\omega \geq 3$.

\[
\frac{\omega}{\delta} \geq \frac{n}{n - \left\lceil \frac{n}{\omega} \right\rceil} = \frac{n}{n - \frac{n}{\omega}} = \frac{\omega^2}{n(\omega - 1)} \geq \frac{9}{2n}.
\]

If $n \geq 9$ the bound is proved. If $n < 9$, in this case, and due to the fact that all odd number less than 9 are primes, necessarily $\omega = n$ and the corresponding graphs are $K_5$ and $K_7$ ($K_3$ is excluded by hypothesis). These two graphs satisfy the inequality.

Now, let us characterize the extremal graphs in each of the above cases.

(i) If $n = kw + l$, where $1 \leq l \leq \omega - 1$, or $n$ is even, equality holds if and only if
\[
\left\lceil \frac{n}{\omega} \right\rceil \left( n - \left\lceil \frac{n}{\omega} \right\rceil \right) = \left\lceil \frac{n}{2} \right\rceil \left( n - \left\lceil \frac{n}{2} \right\rceil \right) \quad \text{and} \quad \omega \left\lceil \frac{n}{\omega} \right\rceil = 2 \left\lceil \frac{n}{2} \right\rceil
\] (5)
a) If \( n \) is even, the second equality of (5) implies that \( \omega \lfloor n/\omega \rfloor = n \), so \( l = 0 \), and then the first equality of (5) implies that \( \omega = 2 \). Thus, \( \delta = n/2 \) and \( G \) is necessarily the balanced complete bipartite \( T_2(n) \).

b) If \( n = k\omega - l \) with \( 1 \leq l \leq \omega + 1 \), then \( \omega \lceil n/\omega \rceil = n + 1 \), which, using the second equality of (5) implies that \( n \) is odd. Thus the first equality of (5) implies that \( k = (n - 3)/2 \) or \( k = (n - 1)/2 \). The case \( k = (n - 3)/2 \) leads to \( \omega < 2 \), so it is excluded, and then necessarily \( k = (n - 1)/2 \) which leads to \( \delta = (n - 1)/2 \). Thus by Lemma 2.1 \( G \) is the balanced complete bipartite graph \( T_2(n) \) or the cycle \( C_5 \).

\( (ii) \) If \( n = k\omega \) and \( n \) is odd (in which case \( \omega \) is also odd), the equality leads to

\[
\frac{\omega}{n - \lfloor n/\omega \rfloor} = \frac{\omega}{n - \frac{n}{\omega}} = \frac{\omega^2}{n(\omega - 1)} = \frac{2}{n - \lfloor \frac{n}{\omega} \rfloor} = \frac{4}{n - 1}
\]

which is true if and only if \( \omega = 2\sqrt{n}/(\sqrt{n} - 1) \) or \( \omega = 2\sqrt{n}/(\sqrt{n} + 1) \). Since \( \omega \) is at least 3 and integer, necessarily \( \omega = 3 \) and \( n = 9 \), and then using Theorem 1.1, \( G \) is the balanced complete 3-partite graph \( T_3(9) \).

The upper bound can be proved exactly as the upper bound of Theorem 2.1.

\[ \square \]

### 2.2. Clique number and average degree

As in the case of minimum degree, AGX 2 obtained and proved automatically the lower and upper bounds on \( \omega + \bar{d} \) and on \( \omega \cdot \bar{d} \). Again, they are given in the summary of results table in the Appendix. The lower and upper bounds on \( \omega - \bar{d} \) and \( \omega / \bar{d} \) are proved below.

**Théorème 2.3 :** Let \( G = (V, E) \) be a connected graph on \( n \) vertices with clique number \( \omega \) and average degree \( \bar{d} \). Then

\[
\omega - \bar{d} \geq \begin{cases} 
2k^2 - k & \text{if } n = k^2, \\
3k + 1 - n - \frac{k^2(k+1)}{n} & \text{if } k^2 \leq n \leq k(k + 1), \\
3k + 2 - n - \frac{k(k+1)^2}{n} & \text{if } k(k + 1) \leq n \leq (k + 1)^2 - 1,
\end{cases}
\]

where \( k = \lfloor \sqrt{n} \rfloor \). Equality holds if and only if \( G \) is the balanced complete multipartite graph \( T_k(n) \) for \( k^2 \leq n < k(k + 1) \), if and only if \( G \)
is $T_k(n)$ or $T_{k+1}(n)$ for $n = k(k + 1)$, if and only if $G$ is $T_{k+1}(n)$ for $k(k + 1) < n \leq (k + 1)^2 - 1$.

$$\omega - d \leq \begin{cases} \frac{n-2}{4} + \frac{2}{n} & \text{if } n \text{ is odd}, \\ \frac{n-2}{4} + \frac{3}{4n} & \text{if } n \text{ is even}. \end{cases}$$

Equality holds if and only if $G$ is composed of a clique on $\omega = \lceil (n + 3)/2 \rceil$ or $\omega = \lfloor (n + 3)/2 \rfloor$ vertices together with $n - \omega$ edges making $G$ connected.

**Proof:**

Lower bound:

If $\omega$ is fixed, then using Theorem 1.1, $\omega - d$ is minimum for a balanced $\omega$-partite graph $T_\omega(n)$ in which an independent set contains $\lceil n/\omega \rceil$ or $\lfloor n/\omega \rfloor$ vertices. Let $r$ (resp. $q$) be the number of independent sets of $T_\omega(n)$ containing $\lceil n/\omega \rceil$ (resp. $\lfloor n/\omega \rfloor$) vertices, with $r + q = \omega$. So $n = \omega \cdot \lceil n/\omega \rceil + q = p \cdot w + q$, with $q < \omega$.

Suppose $q \neq 0$, then $\lceil n/\omega \rceil = p$ and $\lfloor n/\omega \rfloor = p + 1$. For $T_\omega(n)$, we have

$$2m = q(p + 1)(n - p - 1) + p(\omega - q)(n - p).$$

Then

$$\omega - d = \omega - \frac{q(p + 1)(n - p - 1) + p(\omega - q)(n - p)}{n}$$

and simple algebraic manipulations show that

$$\omega - d = \omega + \frac{n}{\omega^2} + \frac{q(\omega - q)}{n\omega} - n = f(\omega) - n. \quad (6)$$

Note that if $n = k^2$ for some integer $k$, the lower bound follows immediately from Theorem 1.1. Thus in the following, we consider only the case $n = k^2 + l$ with $l \neq 0$.

Since the function $g(x) = x + n/x$ is minimum if and only if $x = \sqrt{n}$, it increases for $x \leq \sqrt{n}$, decreases for $x \geq \sqrt{n}$ and $g(x) \geq f(x)$. We need to prove that

$$g(k - 1) \geq f(k) \quad \text{and} \quad g(k + 2) \geq f(k + 1),$$
where \( k = \lceil \sqrt{n} \rceil \). In this case \( n \) can be written \( n = k^2 + l \), where \( 1 \leq l \leq 2k \).

First, we prove that \( g(k - 1) \geq f(k) \). Here we consider two cases.

(i) If \( 1 \leq l \leq k \), we have \( \omega = k, p = k \) and \( q = l \). Then, a substitution in (6) gives

\[
f(k) = k + \frac{n}{k} + \frac{l(k - l)}{nk}
\]

To prove that \( g(k - 1) \geq f(k) \), it is sufficient to prove that

\[
\frac{n}{k - 1} - 1 \geq \frac{n}{k} + \frac{l(k - l)}{nk}
\]  

(7)

The left-hand side of (7) is

\[
\frac{n}{k - 1} - 1 = \frac{k^2 + l}{k - 1} - 1 = \frac{k^2 - k + l + 1}{k - 1} = k + \frac{l + 1}{k - 1}.
\]

The right-hand side of (7) is

\[
\frac{n}{k} + \frac{l(k - l)}{nk} = \frac{k^2 + l}{k} + \frac{l(k - l)}{nk} = k + \frac{l}{k} + \frac{l(k - l)}{nk}.
\]

Thus (7) is equivalent to

\[
\frac{l + 1}{k - 1} \geq \frac{l}{k} + \frac{l(k - l)}{nk}.
\]

For the last term of the above inequality we have

\[
\frac{l(k - l)}{nk} = \frac{lk - l^2}{(k^2 + l)k} \leq \frac{lk}{(k^2 + l)k} \leq \frac{k^2}{(k^2 + l)k} \leq \frac{k^2 + l}{k} = \frac{1}{k}.
\]

Then

\[
\frac{l}{k} + \frac{l(k - l)}{nk} \leq \frac{l}{k} + \frac{1}{k} = \frac{l + 1}{k} \leq \frac{l + 1}{k - 1}.
\]

Therefore (7) is proved in this case.

(ii) If \( k + 1 \leq l \leq 2k \), we have \( \omega = k, p = k + 1 \) and \( q = l - k \). Then, a substitution in (6) gives

\[
f(k) = k + \frac{n}{k} + \frac{(l - k)(2k - l)}{nk}.
\]
As in the case \((i)\), to prove that \(g(k - 1) \geq f(k)\), it is sufficient to prove that

\[
\frac{n}{k - 1} + k - 1 \geq \frac{n}{k} + k + \frac{(l - k)(2k - l)}{n k},
\]

or, equivalently, the following inequality

\[
\frac{l + 1}{k - 1} \geq \frac{1}{k} + \frac{(l - k)(2k - l)}{k(k^2 + l)}.
\]

Take the right-hand side of the above inequality

\[
\frac{l}{k} + \frac{(l - k)(2k - l)}{k(k^2 + l)} \leq \frac{l}{k} + \frac{k^2}{k(k^2 + l)} = \frac{l + 1}{k} \leq \frac{l + 1}{k - 1}.
\]

Therefore (7) is proved in this case.

Now, we prove that \(g(k + 2) \geq f(k + 1)\). Note that in this case, we have \(\omega = k + 1\), so \(p = k\) and \(q = l - k\) or \(p = k - 1\) and \(q = l + 1\).

\((a)\) If \(p = k - 1\) and \(q = l + 1\), it is sufficient to prove that

\[
\frac{n}{k + 2} + k + 2 \geq \frac{n}{k + 1} + k + 1 + \frac{(l + 1)(k - l)}{n(k + 1)}.
\]

The left-hand side of (8) can be written as follows,

\[
\frac{n}{k + 2} + k + 2 = \frac{k^2 + l + 2k + 4}{k + 2} + k = 2k + \frac{l + 4}{k + 2}.
\]

Similarly, the right-hand side can be written as follows,

\[
\frac{n}{k + 1} + k + 1 + \frac{(l + 1)(k - l)}{n(k + 1)} = 2k + \frac{l + 1}{k + 1} + \frac{(l + 1)k - l(l + 1)}{n(k + 1)}.
\]

Now, considering only the last term,

\[
\frac{(l + 1)k - l(l + 1)}{n(k + 1)} \leq \frac{k^2}{n(k + 1)} \leq \frac{k^2}{k^2(k + 1)} = \frac{1}{k + 1}.
\]

Thus to prove (8), it is sufficient to prove that

\[
\frac{l + 2}{k + 1} \leq \frac{l + 4}{k + 2}.
\]
which is equivalent to \( l \leq 4k \). This inequality is true since \( l \leq 2k \).

(b) If \( p = k \) and \( q = l - k \), it is sufficient to prove that
\[
\frac{n}{k + 2} + k + 2 \geq \frac{n}{k + 1} + k + 1 + \frac{l(k + 1 - l)}{n(k + 1)}.
\] (9)

The left-hand side of (9) can be written as follows,
\[
\frac{n}{k + 2} + k + 2 = \frac{k^2 + l + 2k + 4}{k + 2} + k = 2k + \frac{l + 4}{k + 2}.
\]

Similarly, the right-hand side can be written as follows,
\[
\frac{n}{k + 1} + k + 1 + \frac{l(k + 1 - l)}{n(k + 1)} = 2k + \frac{l + 1}{k + 1} + \frac{l(k + 1 - l)}{n(k + 1)}.
\]

Now, considering only the last term of the right-hand side of (9),
\[
\frac{l(k + 1 - l)}{n(k + 1)} \leq \frac{k(k + 1)}{n(k + 1)} = \frac{k}{k^2 + k} \leq \frac{k}{k^2 + k} = \frac{1}{k + 1}.
\]

Thus (9) follows as in (a).

Now, we have to compare \( f(\lceil \sqrt{n} \rceil) = f(k) \) and \( f(\lceil \sqrt{n} \rceil) = f(k + 1) \).

- If \( n = k(k + 1) \), it is easy to see that \( f(k) = f(k + 1) \) and the corresponding extremal graphs are \( T_k(n) \) and \( T_{k+1}(n) \).

- Case \( 1 \leq l \leq k - 1 \), if \( \omega = k \) then \( p = k \) and \( q = l \), if \( \omega = k + 1 \) then \( p = k - 1 \) and \( q = l + 1 \). In this case
\[
f(k) = k + \frac{n}{k} + \frac{l(k - l)}{nk} = 2k + \frac{l}{k} + \frac{l(k - l)}{nk}
\]

and
\[
f(k+1) = k+1 + \frac{n}{k+1} + \frac{(l+1)(k-l)}{n(k+1)} = 2k + \frac{l+1}{k+1} + \frac{(l+1)(k-l)}{n(k+1)}.
\]

Then
\[
f(k + 1) - f(k) = \frac{l+1}{k+1} - \frac{l}{k} + \frac{k-l}{n} \left( \frac{l+1}{k+1} - \frac{l}{k} \right) > 0.
\]

So in this case the extremal graph corresponds to \( T_k(n) \).
Case \( l > k \), if \( \omega = k \) then \( p = k + 1 \) and \( q = l - k \) and if \( \omega = k + 1 \) then \( p = k \) and \( q = l - k \). In this case we have
\[
f(k) = k + \frac{n}{k} + \frac{(l - k)(2k - l)}{nk} = 2k + 1 + \frac{l - k}{k} + \frac{(l - k)(2k - l)}{nk}.
\]
Similarly, we have
\[
f(k + 1) = 2k + 1 + \frac{l - k}{k + 1} + \frac{(l - k)(2k - l)}{n(k + 1)}.
\]
Then
\[
f(k + 1) - f(k) = \left( l - k \right) + \frac{(l - k)(2k - l)}{n} \left( \frac{1}{k + 1} - \frac{1}{k} \right) < 0.
\]
So in this case the extremal graph corresponds to \( T_{k + 1}(n) \).

**Upper bound:**
For given order \( n \), if the clique number \( \omega = t \) is fixed, then
\[
m \geq \frac{t(t - 1)}{2} + n - t
\]
with equality if and only if \( G \) is composed of a clique on \( t \) vertices together with \( n - t \) edges making \( G \) connected. So
\[
\omega - \overline{d} \leq t - \frac{t(t - 1) + 2n - 2t}{n} = \frac{(n + 3)t - t^2 - 2n}{n}
\]
This last expression, as a function \( f(t) \) of \( t \), reaches its maximum for \( t = \left\lfloor \frac{n + 3}{2} \right\rfloor \) or \( t = \left\lceil \frac{n + 3}{2} \right\rceil \). Substitutions show that
\[
f \left( \left\lfloor \frac{n + 3}{2} \right\rfloor \right) = f \left( \left\lceil \frac{n + 3}{2} \right\rceil \right) = \left\{ \begin{array}{ll} \frac{n - 2}{4} + \frac{2}{n} & \text{if } n \text{ is odd,} \\ \frac{n - 2}{4} + \frac{2}{4n} & \text{if } n \text{ is even.} \end{array} \right.
\]
Thus the result follows. \( \square \)

**Théorème 2.4:** Let \( G = (V, E) \) be a simple graph with at least one edge. Then
\[
\frac{n}{\left\lfloor \frac{n}{2} \right\rfloor} \leq \frac{\omega}{d} \leq \frac{n \cdot t}{t^2 - 3t + 2n}, \quad (10)
\]
where
\[
t = \begin{cases} 
\lfloor \sqrt{2n} \rfloor & \text{if } n = k^2 + l \text{ and } 0 \leq l \leq k, \\
\lceil \sqrt{2n} \rceil & \text{if } n = k^2 + l \text{ and } k \leq l \leq 2k.
\end{cases}
\]

The lower bound is attained if and only if \( G \) is the balanced complete bipartite graph \( T_2(n) \) or \( K_3 \), and the upper bound if and only if \( G \) is composed of a clique on \( t \) vertices together with \( n - t \) edges making \( G \) connected.

**Proof :**

**Lower bound :** We consider four cases:

(i) If \( n \) is even, the first inequality in (10) reduces to \( d \leq \omega \cdot n/4 \) or equivalently \( m \leq \omega \cdot n^2/8 \). Using Theorem 1.1, it suffices to show that

\[
\left( 1 - \frac{1}{\omega} \right) \cdot \frac{n^2}{2} \leq \frac{\omega^2}{4} \cdot \frac{n^2}{2}.
\]

This inequality being equivalent to \( (\omega - 2) \geq 0 \) is true and equality holds if and only if \( \omega = 2 \). Thus, using again Theorem 1.1, equality for the lower bound in (10) holds if and only if \( G \) is \( T_2(n) \).

(ii) If \( n \) is odd and \( \omega \geq 4 \), the first inequality in (10) reduces to \( d \leq \frac{\omega}{4} \cdot \frac{n+1}{n} \cdot (n - 1) \). Since \( d \leq n - 1 \) and \( \frac{\omega}{4} \cdot \frac{n+1}{n} > 1 \), the first inequality in (10) is strict.

(iii) If \( n \) is odd and \( \omega = 3 \), using Theorem 1.1, to show that the lower bound in (10) holds, it is sufficient to observe that

\[
\frac{2}{3} \cdot \frac{n^2}{2} \leq \frac{3}{4} \cdot \frac{n^2 - 1}{2}.
\]

Equality holds if and only if \( n = 3 \) and the corresponding graph is \( K_3 \).

(iv) If \( n \) is odd and \( \omega = 2 \), the result follows immediately from Theorem 1.1.

**Upper bound :** For given order \( n \), if the clique number \( \omega = t \) is fixed, then

\[
m \leq \frac{t(t-1)}{2} + n - t
\]
with equality if and only if $G$ is composed of a clique on $t$ vertices together with $n - t$ edges making $G$ connected. So

$$\omega \leq \frac{n \cdot t}{t^2 - 3t + 2n}.$$  

This last bound, as a function $f(t)$ of $t$ and if we consider $t$ as a real variable, is maximum if and only if $t = \sqrt{2n}$. Thus, since $t$ is integer, $f(t)$ is maximum for $t = \lfloor \sqrt{2n} \rfloor$ or for $t = \lceil \sqrt{2n} \rceil$. Let $k = \lfloor \sqrt{2n} \rfloor$ and $2n = k^2 + l$ with $0 \leq l \leq 2k$, then

$$f(\lfloor \sqrt{2n} \rfloor) = f(k) = \frac{nk}{k^2 - 3k + 2n} = \frac{n}{k - 3 + \frac{2n}{k}} = \frac{n}{2k - 3 + \frac{l}{k}},$$

and similarly

$$f(\lceil \sqrt{2n} \rceil) = f(k + 1) = \frac{n}{2k - 3 + \frac{l+1}{k+1}}.$$  

So

$$f(\lfloor \sqrt{2n} \rfloor) \geq f(\lceil \sqrt{2n} \rceil) \iff \frac{l}{k} \leq \frac{l+1}{k+1} \iff \frac{l}{l+1} \leq \frac{k}{k+1} \iff l \leq k.$$  

Then

$$\max f(t) = \begin{cases}  
    f(\lfloor \sqrt{2n} \rfloor) & \text{if } n = k^2 + l \text{ and } 0 \leq l \leq k, \\  
    f(\lceil \sqrt{2n} \rceil) & \text{if } n = k^2 + l \text{ and } k \leq l \leq 2k. 
\end{cases}$$

and thus the result follows. \hfill\Box

### 2.3. Clique number and maximum degree

When comparing the clique number $\omega$ and the maximum degree $\delta$, AGX 2 proved automatically 6 bounds among 8. They are given in the summary results table in the Appendix. The remaining 2 bounds were obtained as conjectures by AGX 2 and are proved in the following proposition.

**Proposition 2.1**: Let $G = (V, E)$ a connected graph on $n \geq 3$ vertices with clique number $\omega$ and maximum degree $\Delta$. Then

$$\omega - \Delta \leq 1 \text{ and } \frac{\omega}{\Delta} \leq \frac{n}{n - 1}$$

with equality in both cases if and only if $G$ is the complete graph $K_n$.  

Proof:
It is easy to see that if $G$ is $K_n$, equality holds in both cases. To be done, it suffices to prove that the inequalities hold and are strict if $G$ is not complete. Thus assume $G$ is not complete. Due to the connectedness, there must be a vertex $v$ in a maximum clique of $G$ connected to some vertex that does not belong to the clique. So $\Delta \geq d(v) \geq \omega$ and therefore $\omega - \Delta \leq 0$ and $\omega / \Delta \leq 1 < n/(n-1)$.

3. Clique Number and Connectivities

In this section we report on the results obtained by AGX 2 on the comparison of the clique number and the algebraic, node and edge connectivities. We first need a preliminary result.

Lemma 3.1: If $G = (V, E)$ is a complete $k$-partite graph $T_k(n)$ with $k \geq 2$ and independent sets of cardinalities $p_1, p_2, \cdots p_k$, then

$$a = \nu = \kappa = \delta = n - \max_{1 \leq i \leq k} p_i.$$ 

Proof:
It is well known that $a \leq \nu \leq \kappa \leq \delta$ for any graph $G \neq K_n$ [12, 13], so to prove the chain of equalities, the last of which is obvious, we need only prove that

$$a = n - \max_{1 \leq i \leq k} p_i.$$ 

Let $b(G)$ be the largest eigenvalue of the Laplacian of a graph $G$, $a(G)$ its algebraic connectivity, $G_1, \cdots G_l$ its connected components and $\overline{G}$ its complement. It is shown in [12, 13] that

$$a(G) = n - b(\overline{G}) \quad \text{and} \quad b(G) = \max_{1 \leq i \leq l} b(G_i).$$

Thus

$$a(T_k(n)) = n - b(T_k(n)) = n - \max_{1 \leq i \leq k} b(K_{p_i}) = n - \max_{1 \leq i \leq k} p_i,$$

and the result follows.
3.1. Clique number and algebraic connectivity

Using the fact that the path $P_n$ minimizes both the clique number $\omega$ and the algebraic connectivity $\alpha$, AGX proved easily the lower bounds on $\omega + \alpha$ and on $\omega \cdot \alpha$. Similarly, the complete graph $K_n$ maximizes simultaneously $\omega$ and $\alpha$, and AGX proved the upper bounds on $\omega + \alpha$ and on $\omega \cdot \alpha$. The bounds on $\omega - \alpha$ and $\omega / \alpha$, obtained as conjectures, are proved in the next two theorems.

Théorème 3.1 : Let $G = (V, E)$ be a connected graph on $n \geq 3$ vertices, $G \neq K_3$, with clique number $\omega$ and algebraic connectivity $\alpha$. Then

$$\left\lfloor \sqrt{n} \right\rfloor - \left\lfloor \left(1 - \frac{1}{\sqrt{n}}\right)n \right\rfloor = \left\lfloor \sqrt{n} \right\rfloor - \left\lfloor \left(1 - \frac{1}{\sqrt{n}}\right)n \right\rfloor \leq \omega - \alpha \leq n - 2.$$

The lower bound is attained if and only if $G$ is the balanced complete multipartite graph $T_{\left\lfloor \sqrt{n} \right\rfloor}(n)$ or $T_{\left\lceil \sqrt{n} \right\rceil}(n)$, and the upper bound if and only if $G$ is the short kite $KT_{n,n-1}$.

Proof :

The lower bound is an immediate consequence of Lemma 3.1, the lower bound of Theorem 2.1 and the fact that if $G \neq K_n$ then $\alpha \leq \delta$.

For the upper bound, we consider the following three cases:

- If $\omega = n$, $\omega - \alpha = 0$ [12, 13].
- If $\omega = n - 1$, due to connectedness there exists at least one dominating vertex in $G$, and then $a \geq 1$. Let $u_1, u_2, \ldots, u_t$ be dominating vertices in $G$ such that $t \leq \left\lfloor n/2 \right\rfloor$ (even if $G$ contains more dominating vertices) and let $v_1, v_2, \ldots, v_s$ be the remaining vertices ($t + s = n$). Consider the subgraph $H$ of $G$ with vertex set $V$ and edge set composed of all edges of type $u_i v_j$ with $1 \leq i \leq t$ and $1 \leq j \leq s$. From its definition $H$ is the complete bipartite graph $K_{t,s}$ and then by Lemma 3.1, $a(H) = t$. Since $H \subset G$ and they have the same vertex set, according to [12, 13], $a \geq a(H) = t$. Thus

$$\omega - \alpha \leq n - 1 - t \leq n - 2$$

with equality if and only if $G$ contains exactly one dominating vertex, i.e., $G$ is the short kite $KT_{n,n-1}$.
• If $\omega \leq n - 2$, since $a > 0$ for all connected graph, then $\omega - a < n - 2$.

\[\begin{align*}
\text{Theorem 3.2} & : \text{ Let } G = (V, E) \text{ be a connected graph on } n \geq 3 \\
& \text{vertices, } G \neq K_3, \text{ with clique number } \omega \text{ and algebraic connectivity } a. \text{ Then} \\
& \frac{2}{\left\lfloor \frac{n}{2} \right\rfloor} \leq \frac{\omega}{a} \leq n - 1.
\end{align*}\]

The lower bound is attained if and only if $G$ is the balanced complete bipartite graph $T_2(n)$, $K_4$, $K_5$ or $T_3(9)$, and the upper bound if and only if $G$ is the short kite $KT_{n,n-1}$.

\[\begin{align*}
\text{Proof} & : \\
\text{The lower bound is an immediate consequence of Lemma 3.1, the lower bound of Theorem 2.2 and the fact that } a \leq \delta \text{ if } G \neq K_n. \\
\text{The upper bound can be proved exactly as the upper bound of Theorem 3.1.}
\end{align*}\]

\[\begin{align*}
\text{3.2. Clique number and node connectivity}
\end{align*}\]

The comparison of the clique number $\omega$ and the node connectivity $\nu$ produced 4 automated results and 4 conjectures. These conjectures are proved below as corollaries of Theorems 3.1 and 3.2.

\[\begin{align*}
\text{Corollaire 3.1} & : \text{ Let } G = (V, E) \text{ be a connected graph on } n \geq 3 \\
& \text{vertices with clique number } \omega \text{ and node connectivity } \nu. \text{ Then} \\
\left\lfloor \sqrt{n} \right\rfloor - \left\lfloor \left(1 - \frac{1}{\left\lfloor \sqrt{n} \right\rfloor} \right) n \right\rfloor & = \left\lfloor \sqrt{n} \right\rfloor - \left\lfloor \left(1 - \frac{1}{\sqrt{n}} \right) n \right\rfloor \leq \omega - \nu \leq n - 2.
\end{align*}\]

The lower bound is best possible as shown by balanced complete $\left\lfloor \sqrt{n} \right\rfloor$-partite and $\left\lfloor \sqrt{n} \right\rfloor$-partite graphs. The upper bound is attained if and only if $G$ is the short kite $KT_{n,n-1}$ or $K_3$.

\[\begin{align*}
\text{Proof} & : \\
\text{The lower bound is an immediate consequence of Lemma 3.1, the lower}
\end{align*}\]
bound of Theorem 2.1 and the fact that $\nu \leq \delta$.
The upper bound follows from the upper bound of Theorem 2.1 using
the fact that if $G \not\cong K_n$ then $a \leq \nu$.

Corollaire 3.2 : Let $G = (V, E)$ be a connected graph on $n$ vertices
with clique number $\omega$ and node connectivity $\nu$. Then

$$\frac{2}{\left\lfloor \frac{n}{2} \right\rfloor} \leq \frac{\omega}{\nu} \leq n - 1.$$  

The lower bound is attained if and only if $G$ is the balanced complete
bipartite graph $T_2(n)$, $C_5$ or $T_3(9)$, and the upper bound if and only if
$G$ is the short kite $KT_{n,n-1}$.

Proof : The lower bound is an immediate consequence of Lemma 3.1, the lower
bound of Theorem 2.2 and the fact that $\nu \leq \delta$.
The upper bound follows from the upper bound of Theorem 2.2 using
the fact that if $G \not\cong K_n$ then $a \leq \nu$.

3.3. Clique number and edge connectivity

Similarly to the case of $\omega$ and $\nu$, the comparison of the clique num-
er $\omega$ and the edge connectivity $\kappa$ produced 4 automated results and 4
conjectures, which are also proved below as corollaries of Theorems 3.1
and 3.2.

Corollaire 3.3 : Let $G = (V, E)$ be a connected graph on $n$ vertices
with clique number $\omega$ and edge connectivity $\kappa$. Then

$$\left\lfloor \sqrt{n} \right\rfloor - \left\lfloor \left( 1 - \frac{1}{\sqrt{n}} \right) n \right\rfloor = \left\lfloor \sqrt{n} \right\rfloor - \left\lfloor \left( 1 - \frac{1}{\sqrt{n}} \right) n \right\rfloor \leq \omega - \kappa \leq n - 2.$$  

The lower bound is the best possible as shown by balanced complete
$\left\lfloor \sqrt{n} \right\rfloor$-partite and $\left\lceil \sqrt{n} \right\rceil$-partite graphs $T_{\left\lfloor \sqrt{n} \right\rfloor}(n)$ and $T_{\left\lceil \sqrt{n} \right\rceil}(n)$. The
upper bound is obtained if and only if $G$ is a short kite $KT_{n,n-1}$ or $K_3$. 
Proof:
The lower bound is an immediate consequence of Lemma 3.1, the lower bound of Theorem 2.1 and the fact that $\kappa \leq \delta$.
The upper bound follows from the upper bound of Theorem 2.1 using the fact that if $G \neq K_n$ then $\alpha \leq \kappa$. □

Corollaire 3.4: Let $G = (V, E)$ be a connected graph on $n$ vertices with clique number $\omega$ and edge connectivity $\kappa$. Then

$$\frac{2}{n^2} \leq \frac{\omega}{\kappa} \leq n - 1.$$  

The lower bound is attained if and only if $G$ is the balanced complete bipartite graph $T_2(n)$, $C_5$ or $T_3(9)$, and the upper bound if and only if $G$ is the short kite $KT_{n,n-1}$.  

Proof:
The lower bound is an immediate consequence of Lemma 3.1, the lower bound of Theorem 2.2 and the fact that $\kappa \leq \delta$.
The upper bound follows from the upper bound of Theorem 2.2 using the fact that $\kappa \leq \delta$. □

4. Clique number and Index

During the comparison of the clique number $\omega$ and the index $\lambda_1$, the AGX 2 system proved 4 among the 8 bounds, generated conjectures for 2 bounds and extremal graphs only in 2 cases, that correspond to structural conjectures. All the conjectures are proved below.

Proposition 4.1: Let $G = (V, E)$ be a connected graph on $n \geq 3$ vertices with clique number $\omega$ and index $\lambda_1$. Then

$$\omega - \lambda_1 \leq 1$$

with equality if and only if $G$ is the complete graph $K_n$.

Proof:
The result follows from the well known inequality on the chromatic
number and the index $\chi \leq \lambda_1 + 1$, due to Wilf [18], and the fact that $\omega \leq \chi$.

The next proposition is an immediate consequence of a theorem due to Feng, Li and Zhang [11], and the fact that $\omega \leq \chi$. That theorem [11, Theorem 2.2] states that on the class of graphs on $n$ vertices with a given chromatic number $\chi = k$, the index is maximum for the balanced $k$–partite graph $T_k(n)$.

**Proposition 4.2 :** Over all connected graphs on $n \geq 4$ vertices with clique number $\omega$ and index $\lambda_1$, the $k$-partite graphs minimize $\omega - \lambda_1$, where $k = \lfloor \sqrt{n} \rfloor$ or $k = \lceil \sqrt{n} \rceil$.

To prove the lower bound on $\omega/\lambda_1$ we need the following two results.

**Théorème 4.1 ([16]) :** Let $G = (V, E)$ be a connected graph on $n \geq 3$ vertices and $m$ edges with clique number $\omega$ and index $\lambda_1$,

$$\lambda_1 \leq \sqrt{2m \cdot \frac{\omega - 1}{\omega}}.$$  

**Lemma 4.1 :** Let $G = (V, E)$ be a connected graph on $n \geq 3$ with clique number $\omega$ and index $\lambda_1$, such that $n = k\omega + l$ with $0 \leq l \leq \omega - 1$. Then

$$\lambda_1 \leq \sqrt{((\omega - l) \left\lfloor \frac{n}{\omega} \right\rfloor (n - \left\lfloor \frac{n}{\omega} \right\rfloor) + l \left\lceil \frac{n}{\omega} \right\rceil (n - \left\lceil \frac{n}{\omega} \right\rceil)} \cdot \frac{\omega - 1}{\omega}. \tag{11}$$

**Proof :** Since the square root is an increasing function and, for $\omega$ fixed, the size $m$ of a graph is maximum for the balanced complete $\omega$-partite graph, the upper bound in Theorem 4.1 is maximum for the balanced complete $\omega$-partite graph. For this graph we have

$$2m = \sum_{v \in V} d(v) = (\omega - l) \left\lfloor \frac{n}{\omega} \right\rfloor \left( n - \left\lfloor \frac{n}{\omega} \right\rfloor \right) + l \left\lceil \frac{n}{\omega} \right\rceil \left( n - \left\lceil \frac{n}{\omega} \right\rceil \right).$$

Thus the bound follows.
Théorème 4.2 : Let $G = (V, E)$ be a connected graph on $n \geq 3$ vertices with clique number $\omega$ and index $\lambda_1$. Then

$$\frac{\omega}{\lambda_1} \geq \frac{2}{\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}}$$

with equality if and only if $G$ is the balanced complete bipartite graph $T_2(n)$.

Proof:

We consider three cases, depending on the values of $\omega$.

(i) If $\omega \geq 4$ since $\lambda_1 \leq \lambda_1(K_n) = n - 1$ we have

$$\frac{\omega}{\lambda_1} \geq \frac{4}{n - 1} > \frac{2}{\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}}.$$ 

(ii) If $\omega = 3$, we have three subcases corresponding on the values of $n \mod 3$.

If $n = 3k$, from (11) we have

$$\lambda_1 \leq \sqrt{2 \cdot \frac{n}{3} \cdot \left( n - \frac{n}{3} \right)} = \frac{2n}{3}.$$

Then easy computations show that for all $n \geq 3$,

$$\frac{\omega}{\lambda_1} \geq \frac{9}{2n} > \frac{2}{\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}}.$$ 

If $n = 3k + 1$, from (11) we have

$$\lambda_1 \leq \sqrt{2 \cdot \frac{n - 1}{3} \cdot \left( n - \frac{n - 1}{3} \right) + \frac{n + 2}{3} \cdot \left( n - \frac{n + 2}{3} \right) \cdot \frac{2}{3}} = \frac{2}{3} \sqrt{n^2 - 1}.$$

So

$$\frac{\omega}{\lambda_1} \geq \frac{9}{2 \sqrt{n^2 - 1}} > \frac{2}{\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}}.$$
If \( n = 3k + 2 \), from (11) we have

\[
\lambda_1 \leq \sqrt{\frac{n - 2}{3} \cdot \left( n - \frac{n - 2}{3} \right) + 2 \cdot \frac{n + 1}{3} \cdot \left( n - \frac{n + 1}{3} \right) \cdot \frac{2}{3} = \frac{2}{3} \sqrt{n^2 - 1}}.
\]

So

\[
\frac{\omega}{\lambda_1} \geq \frac{9}{2\sqrt{n^2 - 1}} > \frac{2}{\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}}.
\]

(iii) If \( \omega = 2 \), using (11) we get the bound. Equality holds if and only if \( m \) is maximum and by Theorem 1.1, the corresponding graph is the balanced complete bipartite one.

Recall the following two theorems.

**Théorème 4.3 ([7])** : For \( n > k \geq 4 \), we have

\[
k - 1 \leq \lambda_1(KT_{n,k}) < \frac{1}{2} \left( k - 3 + \sqrt{(k+1)^2 + 4} \right) < k.
\]

**Théorème 4.4 ([11])** :

– Among all graphs on \( n \) vertices with \( \chi = 2 \), the path \( P_n \) minimizes \( \lambda_1 \).

– Among all graphs with an odd number \( n \) of vertices and \( \chi = 3 \), the cycle \( C_n \) minimizes \( \lambda_1 \).

– Among all graphs with an even number \( n \) of vertices and \( \chi = 3 \), the graph composed of a cycle \( C_{n-1} \) together with a pending edge minimizes \( \lambda_1 \).

– Among all graphs on \( n \) vertices with \( \chi \geq 4 \), the kite \( KT_{n,\chi} \) minimizes \( \lambda_1 \).

**Proposition 4.3** : Over all connected graphs on \( n \geq 4 \) vertices with clique number \( \omega \) and index \( \lambda_1 \), the long kite \( KT_{n,3} \) maximizes \( \omega/\lambda_1 \).

**Proof** : Let \( G \) be a graph on \( n \) vertices with index \( \lambda_1 \), clique number \( \omega \) and
chromatic number $\chi$.
If $\omega = 2$, then $\lambda_1$ is minimum for the path $P_n$.
If $\omega = 3$, then the girth of $G$ is $g = 3$. It is proved in [5] that for $g = 3$, $\lambda_1$ is minimum for the long kite $KT_{n,3}$.
If $\omega \geq 4$ and $\omega = \chi$, then from Theorem 4.4, $\lambda_1$ is minimum for the kite $KT_{n,\omega}$.
If $\omega \geq 4$ and $\omega < \chi$ (recall that $\omega \leq \chi$), from Theorem 4.4 and Theorem 4.3, $\lambda_1 > \lambda_1(KT_{n,\chi}) > \lambda_1(KT_{n,\omega})$.
From the above cases, necessarily $\omega/\lambda_1$ reaches its maximum for a kite. Thus, it remains to prove that the maximum is attained for $KT_{n,3}$ among the set of kites.
First, we compare the values corresponding to $P_n = KT_{n,2}$ and $KT_{n,3}$ for $n \geq 4$. On one hand,

$$\frac{\omega(P_n)}{\lambda_1(P_n)} = \frac{1}{\cos \left( \frac{\pi}{n+1} \right)} \leq \frac{1}{\cos \left( \frac{\pi}{5} \right)}.$$

On the other hand, and since $\lambda_1(KT_{n,3}) < \sqrt{5}$ (see [5]),

$$\frac{\omega(KT_{n,3})}{\lambda_1(KT_{n,3})} > \frac{3}{\sqrt{5}}.$$ 

Therefore

$$\frac{\omega(P_n)}{\lambda_1(P_n)} < \frac{\omega(KT_{n,3})}{\lambda_1(KT_{n,3})}.$$

Now, we compare the values corresponding to $KT_{n,3}$ and $KT_{n,\omega}$ for $\omega \geq 4$. From Theorem 4.3, we have

$$\frac{\omega(KT_{n,\omega})}{\lambda_1(KT_{n,\omega})} = \frac{\omega}{\lambda_1(KT_{n,\omega})} \leq \frac{\omega}{\omega - 1}.$$

Easy calculations show that, for all $\omega \geq 4$,

$$\frac{\omega}{\omega - 1} < \frac{3}{\sqrt{5}} < \frac{\omega(KT_{n,3})}{\lambda_1(KT_{n,3})}.$$

This completes the proof.
5. Clique Number and Metric Invariants

5.1. Clique number and average distance

Using the fact that the path $P_n$ minimizes $\omega$ and maximizes $\overline{l}$, AGX 2 found and proved the lower bounds on $\omega - \overline{l}$ and on $\omega / \overline{l}$. Similarly, using the fact that the complete graph $K_n$ maximizes $\omega$ and minimizes $\overline{l}$, it found and proved the upper bounds on $\omega - \overline{l}$ and on $\omega / \overline{l}$. In 3 among the 4 other cases, conjectures were obtained and are proved in the next 2 theorems. In the remaining case, which corresponds to the upper bound on $\omega \cdot \overline{l}$, only extremal graphs (kites) were provided by the system. An implicit solution is given in that case.

Théorème 5.1 : Let $G = (V, E)$ be a connected graph on $n \geq 3$ vertices with clique number $\omega$ and average distance $\overline{l}$. Then

$$\begin{align*}
&\begin{cases}
\frac{7}{2} - \frac{1}{2n} - \frac{1}{2m} - 2 & \text{if } n \text{ is even}, \\
\frac{7}{2} - \frac{1}{2n} & \text{if } n \text{ is odd}, 
\end{cases} \\
&\frac{1}{2} \leq \omega + \overline{l} \leq n + 1.
\end{align*}$$

The lower bound is attained if and only if $G$ is the balanced complete bipartite graph $T_2(n)$, and the upper bound if and only if $G$ is the complete graph $K_n$.

Proof :
Lower bound : Since $\overline{l} \geq 1$, if $\omega \geq 3$ we have

$$\omega + \overline{l} \geq 4 + \frac{m + 2 \left( \frac{n(n-1)}{2} - m \right)}{\frac{n(n-1)}{2}} = 4 - \frac{2m}{n(n-1)}.$$

Hence, suppose that $\omega = 2$. It follows that

$$\omega + \overline{l} \geq 2 + \frac{m + 2 \left( \frac{n(n-1)}{2} - m \right)}{\frac{n(n-1)}{2}} = 4 - \frac{2m}{n(n-1)}.$$

Then using Theorem 1.1, we have

$$\omega + \overline{l} \geq 4 - \frac{2 \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil}{n(n-1)} = \begin{cases}
\frac{7}{2} - \frac{1}{2n} - \frac{1}{2m} & \text{if } n \text{ is even}, \\
\frac{7}{2} - \frac{1}{2n} & \text{if } n \text{ is odd},
\end{cases}$$
with equality if and only if $G$ is the balanced complete bipartite graph.

Upper bound: If $G \equiv K_n$, we have the result. If $G$ is not complete, using the relation $[6] \Delta + \bar{l} \leq n + 1 - 2/n$ (with equality if and only if $G$ is the star $S_n$) and the fact that $\omega \leq \Delta$, we have

$$\omega + \bar{l} \leq \Delta + \bar{l} \leq n + 1 - \frac{2}{n} < n + 1.$$

This completes the proof.

\[\square\]

**Théorème 5.2**: Let $G = (V, E)$ be a connected graph on $n \geq 3$ vertices with clique number $\omega$ and average distance $\bar{l}$. Then

$$\omega \cdot \bar{l} \geq \begin{cases} 3 - \frac{1}{n+1} & \text{if } n \text{ is even,} \\ 3 - \frac{1}{n} & \text{if } n \text{ is odd,} \end{cases}$$

with equality if and only if $G$ is the balanced complete bipartite graph $T_2(n)$.

**Proof:**

This result can be proved exactly as the lower bound of Theorem 5.1. $\square$

AGX 2 found no conjecture about the upper bound on $\omega \cdot \bar{l}$, except the fact that it must be reached for some kite. Indeed, it is easy to see that for $\omega$ fixed, $\bar{l}$ is maximum for a the kite $KT_{n, \omega}$. Note that for a fixed order $n$ and clique number $\omega$, the average distance of a kite $KT_{n, \omega}$ is given by

$$\bar{l}(n, \omega) = \frac{2\omega^3 - (3n + 6)\omega^2 + (9n + 4)\omega}{3n(n-1)} + \frac{n + 1}{3} - \frac{2}{n - 1}.$$

Thus the maximum of $\omega \cdot \bar{l}$ among all connected graph on $n \geq 4$ vertices is reached for the value $\omega^*$ that maximizes the expression

$$\frac{2\omega^4 - (3n + 6)\omega^3 + (9n + 4)\omega^2}{3n(n-1)} + \frac{(n + 1)\omega}{3} - \frac{2\omega}{n - 1}.$$
5.2. Clique number and remoteness

Proposition 5.1: Let $G = (V, E)$ be a connected graph on $n \geq 3$ vertices with clique number $\omega$ and remoteness $\rho$. Then

$$
\begin{align*}
\text{if } n \text{ is even, } & \quad \frac{7}{2} - \frac{1}{2n-2} \\
\text{if } n \text{ is odd, } & \quad \frac{7}{2} - \frac{1}{2n}
\end{align*}
$$

\leq \omega + \rho \leq n + 1.

The lower bound is attained if and only if $G$ is the balanced complete bipartite graph $T_2(n)$, and the upper bound if and only if $G$ is the complete graph $K_n$.

Proof: The lower bound follows immediately from the lower bound of Theorem 5.1 and the relation $\rho \geq l$.

For the upper bound, note that

(i) From the definitions, $\rho \leq D$ with equality if and only if $G$ is $K_n$.

(ii) $\omega + D \leq n + 1$, since a diametric path contains at most 2 vertices from a clique of $G$, i.e., $D \leq n - \omega + 1$.

Thus the result follows.

Proposition 5.2: Let $G = (V, E)$ be a connected graph on $n \geq 3$ vertices with clique number $\omega$ and remoteness $\rho$. Then

$$
\omega \cdot \rho \geq \begin{cases} 
3 & \text{if } n \text{ is even,} \\
3 - \frac{1}{n-1} & \text{if } n \text{ is odd,}
\end{cases}
$$

with equality if and only if $G$ is the balanced complete bipartite graph $T_2(n)$;

$$
\omega \cdot \rho \leq \max_{i=1,2} \frac{-\omega_i^3 + \omega_i^2 + (n(n - 1) - 2)\omega_i}{2(n - 1)},
$$

where $\omega_1 = 1 + \left[ \left( \sqrt{3n^2 - 3n + 3} \right) / 3 \right]$ and $\omega_2 = 1 + \left[ \left( \sqrt{3n^2 - 3n + 3} \right) / 3 \right]$, with equality if and only if $G$ is the kite $KT_{n,\omega_1}$ or $KT_{n,\omega_2}$.

Proof: Lower bound: If $\omega \geq 3$, since $\rho \geq 1$ the bound is not reachable. So
suppose that $\omega = 2$, let $v$ be a vertex of minimum degree in $G$ and note $\rho(v)$ the normalized transmission of $v$. Then

$$\rho \geq \rho(v) \geq \frac{\delta + 2(n - \delta - 1)}{n - 1} = 2 - \frac{\delta}{n - 1}.$$ 

Now, using the upper bound on $\delta$ deduced from Theorem 1.1 (as in the proofs of Theorem 2.1 and 2.2),

$$\rho \geq 2 - \frac{\lfloor \frac{n}{2} \rfloor}{n - 1}$$

with equality if and only if $G$ is $T_2(n)$. Thus the result follows.

Upper bound: It is easy to see that for fixed order $n$ and clique number $\omega$, the kite $KT_{n,\omega}$ maximizes the remoteness $\rho$, and we have

$$\omega \cdot \rho(KT_{n,\omega}) = \frac{-\omega^3 + \omega^2 + (n(n - 1) - 2)\omega}{2}.$$ 

The last expression, considered as a function of a continuous variable $\omega$ is maximum (using the derivative) for $\omega = 1 + (\sqrt{3n^2 - 3n + 3})/3$. Since $\omega$ is an integer, the maximum is reached for $\omega = 1 + \lfloor (\sqrt{3n^2 - 3n + 3})/3 \rfloor$ or $\omega = 1 + \lceil (\sqrt{3n^2 - 3n + 3})/3 \rceil$.

Note that, the maximum is reached when $\omega = \omega_1$ for some values of $n$ such as $n = 8$, when $\omega = \omega_2$ for some other values of $n$ such as $n = 9$, and when $\omega = \omega_1$ as well as when $\omega = \omega_2$ for some other values of $n$ such as $n = 10$. \hfill $\square$

### 5.3. Clique number and radius

When comparing the clique number $\omega$ and the radius $r$, 6 bounds among 8 were proved automatically and AGX 2 provided 2 conjectures proved in the following theorem.

**Théorème 5.3**: Let $G = (V, E)$ be a connected graph on $n \geq 3$ vertices with clique number $\omega$ and radius $r$. Then

$$\omega + r \leq n + 1 \text{ and } \omega \cdot r \leq \left(n - 2 \left\lfloor \frac{n}{4} \right\rfloor \right) \cdot \left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right).$$

Both bounds are best possible as shown by the complete graph $K_n$ for the sum, and by the kite $KT_{n,\omega}$ where $\omega = n - 2 \left\lfloor \frac{n}{4} \right\rfloor$ for the product.
Proof:
The bound on the sum follows from \( \omega + D \leq n + 1 \) (see the proof of Proposition 5.1) and the fact that \( r \leq D \). Also, it is easy to check that the bound is attained for the complete graph \( K_n \), among others.

For the bound on the product, it is obvious that equality holds for the kite \( KT_{n,\omega} \) where \( \omega = n - 2 \left\lfloor \frac{n}{4} \right\rfloor \). So we have to prove the inequality. A graph \( G \) with radius \( r \) must contain a path \( P \) of length \( 2r - 1 \) (see [10]) and a clique in \( G \) contains at most 2 vertices from \( P \). Thus

\[
\omega \cdot r \leq (n - 2r + 2) \cdot r = (n + 2)r - 2r^2.
\]

It is easy (e.g. using derivatives) to see that this last formula is maximum only for \( r = \lceil (n + 2)/4 \rceil \) or \( r = \lfloor (n + 2)/4 \rfloor \). Substitutions of \( r \) by its possible values show that the maximum of that formula is reached:

- only for \( r = \lfloor n/4 \rfloor + 1 \) if \( n \) is odd \( (n = 4k + 1 \) or \( n = 4k + 3) \);
- for \( r = \lfloor n/4 \rfloor \) or \( r = \lfloor n/4 \rfloor + 1 \) if \( n \) is even \( (n = 4k \) or \( n = 4k + 2) \).

\[ \Box \]

5.4. Clique number and diameter

When comparing the clique number \( \omega \) and the diameter \( D \), as in the case of the radius 6 bounds among 8 were found and proved automatically by AGX 2. Moreover this system provided 2 conjectures next proved.

**Proposition 5.3**: Let \( G = (V, E) \) be a connected graph on \( n \geq 3 \) with clique number \( \omega \) and diameter \( D \). Then

\[
\omega + D \leq n + 1 \quad \text{and} \quad \omega \cdot D \leq \left\lfloor \frac{n + 1}{2} \right\rfloor \left\lceil \frac{n + 1}{2} \right\rceil.
\]

Both bounds are best possible as shown by the path \( P_n \), the kite \( KT_{n,\omega} \) and other graphs for the lower bound, and the kites \( KT_{n,\lfloor n/2 \rfloor - 1} \) and \( KT_{n,\lceil n/2 \rceil} \) among other graphs for the upper bound.

Proof:
The first inequality has already been proved (see again proof of Proposition 5.1). The second inequality is a direct consequence of the first one.
It is straightforward to check that the bounds are tight for the graphs mentioned.

6. Conclusion

Using the system AGX 2, a systematic comparison of the clique number with other graph invariants, i.e., minimum, average and maximum degree, algebraic, node and edge connectivity, index, average distance, remoteness, radius and diameter, has been made. For each invariant $i$, eight bounds have been sought for, i.e., lower and upper bounds for an expression of the form $\omega \oplus i$ where $\oplus$ belongs to \{+, −, ×, ÷\}. In 50 of 88 cases, simple best possible bounds could be found and proved automatically. In the remaining 38 cases, conjectures were obtained either as (i) both a structural result, i.e., description of extremal graphs, and an algebraic relation (17 cases), (ii) a structural result from which a relation was derived by hand (17 cases) or (iii) a structural result from which no relation was, as yet, obtained (4 cases). Of these 38 conjectures, 1 was known, 36 are proved in this paper and an implicit solution is given for the remaining case.

Références


Appendix

The following table summarizes results about all bounds on $\omega \oplus i$ for the invariants considered in this paper. We provide, when available, the algebraic formula together with the family(ies) of extremal graphs for each bound. In the status (st.) column (the first one for the lower bound and last one for the upper bound), we refer to the proposition (P#), corollary (C#) or theorem (T #) that contains the corresponding result in this paper. When the result is immediate, we refer to that using “Im.” In order to simplify the formulas presented in the table we use the following notations:

$$ p = \left\lfloor \frac{n}{2} \right\rfloor, \quad q = \left\lceil \frac{n}{2} \right\rceil, \quad s = \left\lfloor \frac{n+1}{2} \right\rfloor, \quad l_1 = \left\lfloor \frac{n+1}{2} \right\rfloor, \quad l_2 = \left\lceil \frac{n+1}{2} \right\rceil, \quad \omega_1 = 1 + \frac{\sqrt{3n^2-3n+3}}{3}, \quad \omega_2 = 1 + \frac{\sqrt{3n^2-3n+3}}{3}. \quad F(\omega) = -\omega^3 + \omega^2 + (n^2-n-2)\omega,$$

and

$$ t = \begin{cases} \lceil 2n \rceil & \text{if } n = k^2 + l \text{ and } 0 \leq l \leq k; \\ \lfloor 2n \rfloor & \text{if } n = k^2 + l \text{ and } k \leq l \leq 2k. \end{cases} $$

Table 1. Details of conjectures obtained with AGX2

<table>
<thead>
<tr>
<th>st.</th>
<th>G for $p_{\text{min}}$</th>
<th>$p_{\text{min}}$</th>
<th>$t_2 \oplus t_2$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>G for $\omega_3$</th>
<th>st.</th>
</tr>
</thead>
<tbody>
<tr>
<td>T 2.1</td>
<td>$k$-partite</td>
<td>$k - \left\lfloor \frac{n}{2} \right\rfloor$</td>
<td>$\omega - \delta$</td>
<td>$n - 2$</td>
<td>$KT_{n, n-1}$</td>
<td>T 2.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Im.</td>
<td>Tree</td>
<td>3</td>
<td>$\omega + \delta$</td>
<td>$2n-1$</td>
<td>$N_{\omega}$</td>
<td>Im.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T 2.2</td>
<td>$K_3 \cup K_{n, n}$, $K_{n, n}$</td>
<td>$\frac{2}{\pi}$</td>
<td>$\omega / \nu$</td>
<td>$n - 1$</td>
<td>$KT_{n, n-1}$</td>
<td>T 2.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Im.</td>
<td>Tree</td>
<td>2</td>
<td>$\omega - \eta$</td>
<td>$n(\nu - 1)$</td>
<td>$N_{\nu}$</td>
<td>Im.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T 2.3</td>
<td>$k$-partite</td>
<td>$k - \left\lceil \frac{n}{2} \right\rceil$</td>
<td>$\omega - \Omega$</td>
<td>$\frac{2}{\pi}$</td>
<td>$\frac{2}{\pi}$</td>
<td>$n$ odd</td>
<td>see T 2.3</td>
<td>T 2.3</td>
</tr>
<tr>
<td>Im.</td>
<td>Tree</td>
<td>4</td>
<td>$\omega - \eta$</td>
<td>$2n-1$</td>
<td>$N_{\eta}$</td>
<td>Im.</td>
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</tr>
<tr>
<td>T 2.4</td>
<td>$K_3 \cup K_{p, q}$</td>
<td>$\frac{2}{\pi}$</td>
<td>$\omega / \pi$</td>
<td>$\frac{2}{\pi}$</td>
<td>$n$ even</td>
<td>see T 2.4</td>
<td>T 2.4</td>
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<tr>
<td>Im.</td>
<td>Tree</td>
<td>4</td>
<td>$\omega - \pi$</td>
<td>$n(\nu - 1)$</td>
<td>$N_{\nu}$</td>
<td>Im.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T 3.1</td>
<td>$k$-partite</td>
<td>$k - \left\lceil \frac{n}{2} \right\rceil$</td>
<td>$\omega - \Delta$</td>
<td>$\frac{2}{\pi}$</td>
<td>$\frac{2}{\pi}$</td>
<td>$n$ odd</td>
<td>see T 2.3</td>
<td>T 2.3</td>
</tr>
<tr>
<td>Im.</td>
<td>Tree</td>
<td>4</td>
<td>$\omega - \Delta$</td>
<td>$n(\nu - 1)$</td>
<td>$N_{\nu}$</td>
<td>Im.</td>
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<tr>
<td>T 3.2</td>
<td>$K_3 \cup K_{n, n}$, $T_3(4), K_{n, q}$</td>
<td>$\frac{2}{\pi}$</td>
<td>$\omega / \beta$</td>
<td>$n - 2$</td>
<td>$KT_{n, n-1}$</td>
<td>T 3.2</td>
<td></td>
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</tr>
<tr>
<td>Im.</td>
<td>Tree</td>
<td>4</td>
<td>$\omega - \beta$</td>
<td>$2n-1$</td>
<td>$N_{\beta}$</td>
<td>Im.</td>
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<tr>
<td>C 3.1</td>
<td>$k$-partite</td>
<td>$k - \left\lfloor \frac{n}{2} \right\rfloor$</td>
<td>$\omega - \nu$</td>
<td>$n - 2$</td>
<td>$KT_{n, n-1}$</td>
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<td>Tree</td>
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<td>$\omega - \nu$</td>
<td>$2n-1$</td>
<td>$N_{\nu}$</td>
<td>Im.</td>
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<tr>
<td>C 3.2</td>
<td>$K_3 \cup K_{n, n}$, $K_{n, q}$, $T_3(4), K_{n, q}$</td>
<td>$\frac{2}{\pi}$</td>
<td>$\omega / \alpha$</td>
<td>$n - 1$</td>
<td>$KT_{n, n-1}$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Im.</td>
<td>Tree</td>
<td>2</td>
<td>$\omega - \alpha$</td>
<td>$n(\nu - 1)$</td>
<td>$N_{\alpha}$</td>
<td>Im.</td>
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<td>$k - \left\lfloor \frac{n}{2} \right\rfloor$</td>
<td>$\omega - \kappa$</td>
<td>$n - 2$</td>
<td>$KT_{n, n-1}$</td>
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<td>Tree</td>
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<td>C 3.4</td>
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<td>$\omega - \mu$</td>
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<td>[ K_n ]</td>
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<td>[ K_{p,q} ] [ \frac{n}{n-1} ] [ n \text{ even}, \ n \text{ odd} ] [ \omega + 1 ]</td>
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<td>[ n ]</td>
<td>[ K_n ]</td>
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<tr>
<td>T 5.2</td>
<td>[ K_{p,q} ] [ \frac{1}{n} ] [ n \text{ even}, \ n \text{ odd} ] [ \omega + 1 ]</td>
<td>[ n ]</td>
<td>[ K_n ]</td>
<td>Kite</td>
<td></td>
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</tr>
<tr>
<td>Im.</td>
<td>[ P_n ] [ \omega - \rho ] [ n \text{ even} ]</td>
<td>[ n - 1 ]</td>
<td>[ K_n ]</td>
<td>Im.</td>
<td></td>
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</tr>
<tr>
<td>P 5.1</td>
<td>[ K_{p,q} ] [ \frac{n}{2n-1} ] [ n \text{ even}, \ n \text{ odd} ] [ \omega + \rho ]</td>
<td>[ n + 1 ]</td>
<td>[ K_n ]</td>
<td>P 5.1</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Im.</td>
<td>[ P_n ] [ \omega + \rho ] [ n \text{ even} ]</td>
<td>[ n ]</td>
<td>[ K_n ]</td>
<td>Im.</td>
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<tr>
<td>P 5.2</td>
<td>[ K_{p,q} ] [ \frac{n}{n-4} ] [ n \text{ even} ] [ \omega - \rho ] [ \max { \rho(\omega_1), \rho(\omega_2) } ]</td>
<td>[ n \text{ even} ]</td>
<td>[ K_n ]</td>
<td>P 5.2</td>
<td></td>
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</tr>
<tr>
<td>Im.</td>
<td>[ P_{n,1}, P_{n,2} ] [ 2 - q ] [ n \text{ even} ]</td>
<td>[ n - 1 ]</td>
<td>[ K_n ]</td>
<td>Im.</td>
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<tr>
<td>Im.</td>
<td>[ S_n ] [ 3 ] [ \omega - \rho ] [ n \text{ even} ]</td>
<td>[ n + 1 ]</td>
<td>[ K_n ]</td>
<td>T 5.5</td>
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<td></td>
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</tr>
<tr>
<td>Im.</td>
<td>[ P_{n,1}, P_{n,2} ] [ 2/q ] [ \omega ]</td>
<td>[ n \text{ even} ]</td>
<td>[ K_n ]</td>
<td>Im.</td>
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<tr>
<td>Im.</td>
<td>[ S_n ] [ 2 ] [ \omega ]</td>
<td>[ n \text{ even} ]</td>
<td>[ n \text{ even} ]</td>
<td>[ K_{n,2} ]</td>
<td>T 5.5</td>
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<td></td>
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<tr>
<td>Im.</td>
<td>[ P_{n,1}, P_{n,2} ] [ 3 - n ] [ \omega - \rho ] [ n \text{ even} ]</td>
<td>[ n - 1 ]</td>
<td>[ K_n ]</td>
<td>Im.</td>
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</tr>
<tr>
<td>Im.</td>
<td>[ S_n ] [ 3 ] [ \omega + \rho ] [ n \text{ even} ]</td>
<td>[ n + 1 ]</td>
<td>[ K_n ]</td>
<td>T 5.5</td>
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<tr>
<td>Im.</td>
<td>[ P_{n,1}, P_{n,2} ] [ 3/2 ] [ \omega ]</td>
<td>[ n \text{ even} ]</td>
<td>[ n \text{ even} ]</td>
<td>[ K_n ]</td>
<td>Im.</td>
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<tr>
<td>Im.</td>
<td>[ S_n ] [ 4 ] [ \omega - \rho ] [ n \text{ even} ]</td>
<td>[ n \text{ even} ]</td>
<td>[ K_{n,4} ]</td>
<td>T 5.5</td>
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<tr>
<td>Im.</td>
<td>[ P_{n,1}, P_{n,2} ] [ 4/n ] [ \omega + \rho ] [ n \text{ even} ]</td>
<td>[ n \text{ even} ]</td>
<td>[ K_n ]</td>
<td>Im.</td>
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